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Regular Restrained Domination In GRAPH

M.H. Muddebihal[1], Faimidal Begum[2]

[1]*Department of Mathematics, Gulbarga University, Kalaburgi-585106.*
[2]*Research scholar, Department of Mathematics Gulbarga University, Kalaburgi-585106.*
**Corresponding author's E-mail: muddebihal@gmail.com*

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Abstract

The research of the standard domination variation, particularly regular restrained domination (RRD), is examined in the current paper. Assume that G=(V,E) is a graph. If the induced subgraph ⟨F⟩ is regular, the set F⊆V(G) is a regular restrained dominating set. The minimum cardinality of the regular restrained dominating (RRD) set of a graph G is designated by the notation γ_{rr}(G). We investigated various RRD features and established a relationship with other domination factors.

Keywords: *Graph, Domination number, Restrained domination number, Regular Restrained domination number.*

1. Introduction

The terms from graph theory that isn't covered here can be found in [4]. In this article, finite, undirected, nontrivial, connected graphs without loops or many edges are taken into consideration. The open neighbourhood of $v \in V$ is specified by $\{u \in v / uv \in E\}$ and is indicated by $N(v)$.

subgraph of x and closed neighbourhood is defined by $N[v] = N(v) \cup \{v\}$. We use $\langle x \rangle$ to denote the induced and the maximum degree of vertex in a graph G is denoted by Δ and minimum degree of vertices in $G = (V, E)$ is denoted by $\delta(G)$. The notation $\alpha_0(G)$, $\alpha_1(G)$ is the minimum cardinality of vertices(edges) of a vertex (edge) cover of G and $\beta_0(G)$, $\beta_1(G)$ denotes the minimum number of vertices(edges) in a maximal independent set of vertices(edges) of G.

A set $D \subseteq V$ is a dominating set of G, if every vertex in D is adjacent to a vertex in $V - D$. The domination number of G denoted by $\gamma(G)$ is the minimum cardinality of a minimal dominating set. A indepth study of domination shown in [6,7].

In[3], the concept of restrained domination was introduced and further studied by Zelinkin [11].

A restrained dominating set is a set $F \subseteq V(G)$ where every vertex in $V - F$ is adjacent to a vertex in F and other vertex in $V - F$. The restrained domination number of G , denoted by $\gamma_r(G)$ is the minimum cardinality of restrained dominating set of G.

The goal of this paper is to present and investigate the concept of regular restrained domination and its properties.

A dominating set D of a graph G is a regular restrained dominating set if the induced sub graph $\langle D \rangle$ is regular. The minimum cardinality of regular restrained dominating set is the regular restrained domination number of G and denoted by $\gamma_{rr}(G)$.

2.

RESULTS

The below theorem offers the regular restrained domination for a few standard graphs.

Theorem 1. a) For any graph $G = k_n$ (n≥3) or w_n then $\gamma_{rr}(G)=1$.

b) For any complete bipartite graph $k_{m,n}(m,\ n \geq 2)$, then $\gamma_{rr}(k_{m,n})=2$.

c) For any cycle G= C_{3n} and G=C_{4n} with (n= 1, 2, 3, ……..) , then $\gamma_{rr}(C_{3n})=n$,

$\gamma_{rr}(C_{4n})=2n$.

The proof of the above theorem is simple, hence omit the proof.

Lemma 1: For any path P_{3n+1}(n=1, 2, 3, ………..), $\gamma_{rr}(P_{3n+1})= n+1$ and $\gamma_{rr}(G)$ does not exits for G=P_{2n+1} (n=1,2,4,5,………..), G=P_{4n} (n=2,3,5,6,……….).

Proof: Suppose D is a regular restrained dominating set of P_p (p=3n+1,n=1,2,3,……………), whose vertex set is $V(G) = \{v_1, v_2, v_3, \dots\dots\dots\dots\dots v_p\}$. Since D=$\{v_1, v_4, v_7, v_9, \dots, v_{p-3}, v_p\}$ and each $v_i \in D$, $1 \leq i \leq p$, $deg(v_i)=0$.

Hence it is regular restrained dominating set of G. Then $\forall v_j \in V-D$ is adjacent to atleast one vertex of D and atleast one vertex of $V-D$ and also N[D]=V(G).Hence D is a γ_{rr}-set of G, so that $\gamma_{rr}(P_{3n+1})=n+1$. For the graph $G = P_{2n+1}(n = 1,2,4,5,7,8,\dots\dots.)$, assume D_1 is a regular restrained dominating set and its vertex set is $D_1 = \{v_1,v_4,v_7,v_8,v_9,\dots\dots,v_{2p-1},v_{2p},v_{2p+1}\}$ and for each $v_i \in D_1$,$1\leq i \leq 2p+1$ and the induced subgraph $\langle D_1 \rangle$ is not regular. Hence it is contradiction to the fact. For the graph $G = P_{4n}(n = 2,3,5,6,8,9,\dots\dots)$, assume D_2 is a regular restrained dominating set of $G = P_{4n}$, whose vertex set $D_2 = \{v_1,v_4,v_7,v_8,v_9,\dots\dots,v_{2p-1},v_{2p}\}$ and each $v_i \in D_2$, $1\leq i \leq 2p$, one can easily seen that the induced sub graph $\langle D_2 \rangle$ is not regular. Hence a contradiction. For the graph $G = K_{1,n}$, it is easily verified that γ_{rr} does not exits.

Theorem2: If T is a tree with $n \geq 3$ vertices, then $\gamma_{rr}(T) \geq \Delta(T)$.

Proof: Let B=$\{v_1, v_2, v_3, \dots\dots, v_n\} \subseteq V(T)$ be the set of all non end vertices in T and it exists atleast one vertex u of maximum degree, $deg(u)=\Delta(T)$. Suppose $B_1 = \{v_1, v_2, v_3, \dots\dots, v_m\}$ be the set of all end vertices in T and $B_2 = \{v_1, v_2, v_3, \dots\dots, v_j\} \subseteq B$ be the set of vertices that are not adjacent to B_1. Then $B_1 \cup H$ where $H \subseteq B_2$ forms a minimal restrained dominating set of T. If the induced sub graph $\langle B_1 \cup H \rangle$ is regular, then $\{B_1 \cup H\}$ is a regular restrained dominating set of T. Hence $|B_1 \cup H| \geq \Delta(T)$ which gives $\gamma_{rr}(T) \geq \Delta(T)$.

A dominating set $K \subseteq V(G)$ is a co-regular split dominating set if the induced sub graph $\langle V - K \rangle$ is regular and disconnected. The minimum cardinality of a such a set is called a co-regular split domination number and is denoted by $\gamma_{crs}(G)$,see[10].

If the induced sub graph $\langle D \rangle$ has no edges, the dominating set D of the graph G=(V,E) is an independent dominating set. The lowest cardinality of an independent dominating set is known as the independent domination number i(G) of a graph G. see[1].

The following theorem relates $\gamma_{rr}(G)$ in relation to $\gamma_{crs}(G)$ and $i(G)$.

Theorem 3: To any connected (p, q) graph G, $\gamma_{rr}(G) \leq \gamma_{crs}(G) + i(G)$ and $G \neq K_n (n \geq 3), G \neq W_4$

Proof: Let $V(G) = \{v_1, v_2, v_3, \ldots\ldots, v_n\}$ be the vertex set of G. Now for graph $G = K_n$ with $n \geq 3$, $G = W_4$. Then by definition of γ_s and $\gamma_{crs}(G)$ does not exit. Suppose D=$\{v_1, v_2, v_3, \ldots\ldots, v_p\} \subseteq V(G)$ be a minimal dominating set of G. If $\forall v_i \in D, 1 \leq i \leq n$, deg($v_i$)=0, then D is a minimal independent dominating set of G. Let F=$\{v_1, v_2, v_3, \ldots\ldots, v_m\} \subseteq V(G)$ be the minimal dominating set of G such that the induced sub graph $\langle V(G) - F\rangle$ is regular and which gives more than one component. Then F forms a minimal co-regular split dominating set of G. Suppose M is the set of end vertices in G, then $\{D \cup M\}$ gives a minimal restrained dominating set of G. Since each component of the induced sub graph $\langle D \cup M\rangle$ has same degree, then $\{D \cup M\}$ gives a γ_{rr}-set of G. Further for inequality result of the theorem. We consider following cases.

Case1: Suppose D= $\{D \cup M\}$. Then M= $\{\emptyset\}$. Suppose D$\subset$ $\{D \cup M\}$.Then M $\neq \{\emptyset\}$

Case2: Suppose G has cut vertex $v \in F$ and G contains M, then F$\subset$ $\{D \cup M\}$. If $v \in F$ and G does not contain M, then F= $\{D \cup M\}$. From the above two cases, we have the relation $|D \cup M| \leq |F| + |D|$ which gives $\gamma_{rr}(G) \leq \gamma_{crs}(G) + i(G)$.

The next theorem gives $\gamma_{rr}(G)$ in terms of the number of vertices of G.

Theorem 4: For any connected (p, q) graph G, $\gamma_{rr}(G) \geq \left\lfloor \frac{p}{3} \right\rfloor$. Equality holds for $G = C_{3n}$(n=1,2,3,…).

Proof: Let $V(G) = \{v_1, v_2, v_3, \ldots\ldots, v_k\}$ be the vertex set of G with $|V(G)|$=p and let A_1=$\{v_1, v_2, v_3, \ldots\ldots, v_p\}$ be the set of end vertices in G Let's say D=$\{v_1, v_2, v_3, \ldots\ldots, v_n\} \subseteq V(G) - A_1$ be the smallest set of vertices that encompasses all the vertices in $G - A_1$ such that N[D]=$\{V(G) - A_1\}$. Then D is a minimal dominating set of $G - A_1$. Further $\forall v_m \in \{V(G) - A_1\} - D$, $N(v_m) \neq \emptyset$ and set A_2=$\{A_1 \cup D\}$ gives restrained dominating set of G. If the induced subgraph $\langle A_1 \cup D\rangle$ is regular, then A_2 is regular restrained dominating set of G. Hence, $|A_1 \cup D| \geq \left|\frac{V(G)}{3}\right|$ which gives , $\gamma_{rr}(G) \geq \left\lfloor \frac{p}{3} \right\rfloor$.

The following theorem relates $\gamma_{rr}(G)$ in relation to the number of edges of G and $\Delta(G)$.

Theorem 5: For any connected (p, q) graph G, $\gamma_{rr}(G) \geq \left|\frac{q}{\Delta(G)+1}\right|$, $G \neq K_n (n \geq 5)$, and $G \neq K_{m,n}$(m,n$\geq$ 4).

Proof: Let E=$\{e_1, e_2, e_3, \ldots\ldots\ldots, e_n$ be the set of edges in G with $|E| = q$. Now for the graph $G \neq K_n$ with $n \geq 5$, suppose $n \leq 5$ the $\gamma_{rr}(G)$= $\left\lfloor \frac{q}{\Delta(G)+1} \right\rfloor$=1 and result holds. Further if $n > 5$, $\gamma_{rr}(G) < \left|\frac{q}{\Delta(G)+1}\right|$. Hence $G \neq K_n$ with $n \geq 5$. For graph G=$K_{m,n}$ with m,n$<$ 3, $\gamma_{rr}(G)$=2$<$ $\left\lfloor \frac{q}{\Delta(G)+1} \right\rfloor$, a contradiction. Let B=$\{v_1, v_2, v_3, \ldots\ldots, v_n\} \subset V(G)$ be the set of end vertices in G and $C = V(G) - B$. Then there is a vertex set $H \subseteq C$ such that $\forall v_j \in [V(G) - (H \cup B)]$ is adjacent to atleast one vertex of $\{H \cup B\}$ and atleast one vertex of $V(G) - (H \cup B)$. Then $\{H \cup B\}$ is a γ_r set of G. If the induced sub graph $\langle H \cup B\rangle$ is regular, then $\{H \cup B\}$ is a γ_{rr} set of G. Given that there exists for any graph G at least one vertex u with maximum degree. deg(u)= Δ(G) Thus, it follows $|H \cup B| \geq \left|\frac{E}{\Delta(G)+1}\right|$, which gives $\gamma_{rr}(G) \geq \left|\frac{q}{\Delta(G)+1}\right|$.

The following theorem relates $\gamma_{rr}(G)$ with number of vertices of G and β_0.

Theorem 6: To any connected graph G, $\gamma_{rr}(G) \leq 2(p-\beta_0)$.

Proof: Let $V(G)=\{v_1,v_2,v_3, \ldots\ldots, v_n\}$ be the vertex set of G with $|V(G)|$=p and let V_1=$\{v_1,v_2,v_3, \ldots\ldots, v_k\} \subseteq V(G)$ be the set of vertices such that $dist(u,w) \geq 2$ and $N(u) \cup N(w) = y$, $\forall\, u,w \in V_1$ and $y \in V(G)-V_1$. Clearly $|V_1|=\beta_0(G)$. Further there exists a set D=$\{v_1,v_2,v_3, \ldots\ldots, v_m\} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices of G. Hence D forms γ set of G. Suppose A=$\{v_1,v_2,v_3, \ldots\ldots, v_p\} \subset V(G)$ be the set of end vertices in G and $A_1 = V(G)-A$. Then there exists a vertex set $H \subseteq A_1$ such that $\forall\, v_j \in [V(G)-\{H \cup A\}]$ is adjacent to atleast one vertex of $\{H \cup A\}$ and atleast one vertex of $[V(G)-\{H \cup A\}]$. Then $\{H \cup A\}$ is a γ_r set of G. If the induced sub graph $\langle H \cup A \rangle$ is regular, then $\{H \cup A\}$ is a γ_{rr} set of G. Hence $|H \cup A| \leq 2(|V(G)|-|V_1|)$ which gives $\gamma_{rr}(G) \leq 2(p-\beta_0)$.

A dominating set, If the induced sub graph $\langle D \rangle$ has no isolated vertices, then D is a total dominating set. The minimal cardinality of a total dominating set is called the total domination number $\gamma_t(G)$ of a graph .G , see[2].

A dominating set S is called a perfect dominating set, When each vertex $u \in V-S$, $N(u) \cap S = 1$. The perfect dominating number is denoted by $\gamma_p(G)$, see[8].

Theorem 7: For any connected (p,q) graph G, $\gamma_{rr}(G)+\gamma_p(G) \leq \gamma_t(G)+(p-\alpha_0)$ and $G \neq C_p[p=n+3, n=0,1,2,3 \ldots\ldots$except $G=C_{3n}$ and $G=C_{4n}$ with $n=1,2,3,\ldots\ldots]$.

Proof: Let $V(G)=\{v_1,v_2,v_3, \ldots\ldots\ldots\ldots\ldots, v_n\}$ be the vertex set of G. Now for the graph $G \neq C_p(p=n+3, n=0,1,2,\ldots\ldots\ldots\ldots)$, suppose for $G=C_{3n}$ and $G=C_{4n}$ with $(n=1,2,3,\ldots\ldots)$, $\gamma_{rr}(G)+\gamma_p(G)=2 \leq \gamma_t(G)+(p-\alpha_0)=3$ and $\gamma_{rr}(G)+\gamma_p(G)=4=\gamma_t(G)+(p-\alpha_0)$ result holds. Further if G= $C_p(p=n+3, n=0,1,2 \ldots\ldots)$ except $G=C_{3n}$ and $G=C_{4n}$ with $(n=1,2,3,\ldots\ldots)$, $\gamma_{rr}(G)+\gamma_p(G) > \gamma_t(G)+(p-\alpha_0)$. Hence, $G \neq C_p[p=n+3, n=0,1,2,3 \ldots]$. Suppose $B=\{v_1,v_2,v_3, \ldots\ldots, v_k\} \subseteq V(G)$ be the minimal set of vertices which covers all the edges in G, then $|B|=\alpha_0(G)$. Let D=$\{v_1,v_2,v_3, \ldots\ldots\ldots\ldots\ldots\ldots, v_m\} \subseteq V(G)$ such that N[D]=V(G), then D is a minimal dominating set of G. If the induced sub graph $\langle D \rangle$ has no isolated vertices, then D itself is a total dominating set of G. Otherwise, select $v_j \in \{V(G)-D\}, 1 \leq j \leq n$ and if $\{D\} \cup \{v_j\}$ has no isolated vertices. Clearly $\{D \cup v_j\}$ is a minimal total dominating set of G. Further let $S=\{v_1,v_2,v_3, \ldots\ldots, v_p\} \subseteq V(G)$ be the γ set of G and every vertex $v_i \in V(G)-S$ is adjacent to exactly one vertex of S. Then S is a perfect dominating set of G. Suppose C=$\{v_1,v_2,v_3, \ldots, v_r\} \subset V(G)$ be the set of end vertices in G. Further there is a set $B_1 = \{C\} \cup \{D\}$. Then B_1 forms a minimal restrained dominating set of G. If the induced sub graph $\langle C \cup D \rangle$ is regular, clearly B_1 is a γ_{rr} set of G. It follows that $|B_1|+|S| \leq |D|+(|V(G)|-|B|)$ which gives $\gamma_{rr}(G)+\gamma_p(G) \leq \gamma_t(G)+(p-\alpha_0)$. For the condition $G \neq C_p[p=n+3, n=0,1,2,3 \ldots\ldots$except $G=C_{3n}$ and $G=C_{4n}$ with $n=1,2,3,\ldots\ldots]$ the proof is followed from theorem1.

The following theorem relates γ_{rr}(G) with the number of vertices and edges of a graph G.

Theorem 8: For any connected graph G, γ_{rr}(G)$\leq 2q - p$.

Proof: Let A=$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ be the vertex set of G with $|A|=p$ and let E=$\{e_1, e_2, e_3, \ldots\ldots\ldots, e_n\}$ be the set of edges in G with $|E| = q$. Suppose D be a minimal dominating set of G such that N[D]=A. Further suppose $V_1 = \{v_1, v_2, v_3, \ldots\ldots, v_k\} \subset$A be the set of end vertices in G. Now, $\forall v_j \in \{V(G) - (D \cup V_1)\}$ is adjacent to atleast one vertex of $D \cup V_1$ and at least one vertex of $\{V(G) - (D \cup V_1)\}$. Clearly$\{D \cup V_1\}$ is a $\gamma_r$set of G. If the induced sub graph $\langle D \cup V_1 \rangle$ is regular, then $\{D \cup V_1\}$ is a γ_{rr}set of G. Otherwise, there exists a set $V_2 = \{v_1, v_2, v_3, \ldots, v_p\} \in \{V(G) - (D \cup V_1)\}$ such that the induced subgraph $\langle D \cup V_1 \cup V_2 \rangle$ is regular. Thus $\{D \cup V_1 \cup V_2\}$ is a regular restrained dominating set of G. Hence $|D \cup V_1 \cup V_2| \leq 2|E|$-$|A|$ whih gives γ_{rr}(G)$\leq 2q - p$ The diameter of a connected graph G is the shortest distance that can exist between any two of its furthest vertices.

On a graph G, a function f:V→{0,1,2} satisfies the requirement that each vertex u for which f(u)=0 is adjacent to at least one vertex v for which f(v)=2, which is known as a Roman dominating function. The weight of the Roman dominating function is represented by the formula f(v)=$\sum_{u \in v} f(u)$. The Roman dominating number of a graph G is defined as the smallest weight of the Roman dominating function and is denoted by γ_R(G).

Within the ensuing theorem we relates our concept to γ_R(G) and $i(G)$.

Theorem 9: For any connected graph G, $\gamma_{rr}(G)$+$i(G) \leq diam(G) + \gamma_R$(G) and $G \neq C_p[p = n + 3,\ n = 0,1,2,3 \ldots\ldots$except $G = C_{3n}$ and $G = C_{4n}$ with $n = 1,2,3, \ldots\ldots]$.

Proof: Let V(G)=$\{v_1, v_2, v_3, \ldots\ldots\ldots\ldots\ldots\ldots\ldots, v_k\}$ be the vertex set of G. Now for the graph $G \neq C_p[p = n + 3,\ (n = 0,1,2,3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots)$, suppose for $G = C_{3n}$ and $G = C_{4n}$ with n= $1,2,3, \ldots\ldots$, , $|\gamma_{rr}(G) + i(G)|$=$2n < |diam(G) + \gamma_R(G)| = 3(2n - 1)$ result holds. For the cycles other than $G = C_{3n}$ and $= C_{4n}$, the regular restrained dominating set doesn't exist . Hence $G \neq C_p[p = n + 3, n = 0,1,2,3 \ldots\ldots$Now let A==$\{e_1, e_2, e_3, \ldots\ldots\ldots, e_k\}\subseteq$E(G) be the minimum set of edges which constitutes the shortest distance between any two distinct vertices $u, w \in V(G)$ with dist$(u, w) = diam(G)$. Let B=$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ ⊆V(G) such that $N(v_i) \cap N(v_j) \neq \emptyset$ and every vertex of $V(G) - B$ is adjacent to atleast one vertex of B with N[B]=V(G) which represents B is minimal independent dominating set of G. Let $f: V \rightarrow \{0,1,2\}$ and partition the vertex set V(G) in to (V_0,V_1,V_2) induced by f with $|V_i|$=ni for i=0,1,2 suppose the set V_2 dominates V_0,then $S = V_1 \cup V_2$ forms a minimal Roman dominating set of G. Let $B_1 = \{v_1, v_2, v_3, \ldots\ldots, v_m\}\subset$V(G) be the set of end vertices in G and suppose $B_2 \subset B$ in G so that every vertex of $\{V(G) - (B_1 \cup B_2)\}$ is adjacent to atleast one vertex of $(B_1 \cup B_2)$ and atleast one vertex of $\{V(G) - (B_1 \cup B_2)\}$. Clearly $(B_1 \cup B_2)$ is restrained dominating set of G. If the induced subgraph $\langle B_1 \cup B_2 \rangle$ is regular, then $(B_1 \cup B_2)$ is a γ_{rr} set of G. Hence $|B_1 \cup B_2|$+$|B| \leq |A| + |S|$ which gives $\gamma_{rr}(G)$+ $i(G) \leq diam(G) + \gamma_R$(G).

If the induced subgraph (V-D) contains no isolated vertices, then the dominating set D⊆V(G) is a cototal dominating set. The lowest cardinality of the co total dominating set of G is the co total dominating numberγ_{ct}(G) of G, see[7]

The following theorem relates, $\gamma_{rr}(G)$ with γ_{ct} and the edge domination number, refer [9].

Theorem 10: For each connected graph G, $\gamma_{rr}(G) + \gamma_{ct}(G)\leq P + \delta(G) + \gamma^{1}(G)$.

Proof: Let V(G)= $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ be the vertex set of G with $|V(G)| = p$ and if there exists a vertex v with minimum degree, then $deg(v) = \delta(G)$. Further let E_1=$\{e_1, e_2, e_3, \ldots\ldots\ldots, e_n\}$⊆E(G) be the minimal set of edges which covers all the edges in G. Such that $N[E_1] = E(G)$. Then E_1 forms an edge dominating set of G. Let A=$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ ⊆V(G) be the minimal set of vertices which covers all the vertices in G and A_1=V(G)$-A$. Suppose every vertex in V(G)$-A$ is adjacent to atleast one vertex of A. Then A is dominating set of G and if induced sub graph $\langle V - A\rangle$ does not contain isolated vertices, then A is a γ_{ct} set of G. Let A_2=$\{v_1, v_2, v_3, \ldots\ldots, v_k\}$ ⊆V(G) and further, $\forall\, v_m \in V(G) - A_2$, $N[A_2] = V(G)$ and N(v_m)$\neq \emptyset$, then A_2 forms $\gamma_r$set of G. If the induced subgraph $\langle A_2\rangle$ is regular, then A_2 is γ_{rr}set of G.Hence $|A_2| + |A| \leq |V(G)| + \delta(G) + |E_1|$ which gives $\gamma_{rr}(G) + \gamma_{ct}(G)\leq P + \delta(G) + \gamma^{'}(G)$.

A dominating set F of G is known as split dominating set if the induced sub graph $\langle V - F\rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of split dominating set of G, see [7].

If the induced subgraph $\langle V - K\rangle$ is completely disconnected with at least two vertices, the dominating set K of G is said to be a strong split dominating set of G. The minimum cardinality strong split dominating set of G is represented by the strong split dominance number γ_{ss}(G), see [7].

The subsequent theorem provides the relationship between $\gamma_s(G)$ and γ_{ss}(G) with $\gamma_{rr}(G)$.

Theorem11: For any connected graph G, $\gamma_{rr}(G) + \gamma_{ss}(G)\leq q + \gamma_s(G)$ and $G \neq C_{n+3}$, $(n = 0,1,2,3 \ldots\ldots)$ except $G = C_{3n}$ and $G = C_{4n}$ with $(n = 1,2,3, \ldots\ldots)$

Proof: Let V(G)= $\{v_1, v_2, v_3, \ldots\ldots\ldots\ldots\ldots\ldots, v_k\}$ be the set of vertices in G. Now for the graph $\neq C_{n+3}$ $(n = 0,1,2,3 \ldots\ldots)$, suppose for G=$C_{3n}$ $(n = 2,3,4, \ldots\ldots\ldots)$, and $G = C_{4n}$ with $(n = 1,2,3, \ldots\ldots\ldots\ldots\ldots)$, $\gamma_{rr}(G) + \gamma_{ss} = 2 + 2 < q + \gamma_s(G)$=4+2 and result holds. Further if G=C_5, $\gamma_{rr}(G) + \gamma_{ss}$(G)=5+3> $q + \gamma_s(G) = 5 + 2$. Hence $G \neq C_p[p = n + 3, n = 0,1,2,3 \ldots\ldots\ldots$,]. Let E(G)=$\{e_1, e_2, e_3, \ldots\ldots\ldots, e_n\}$ be the edge set of G with $E(G) = q$. Let $D \subseteq V(G)$ be a minimal dominating be a minimal dominating set of G. If the induced sub graph $\langle V(G) - D\rangle$ has more than one component, then D itself is a split dominating set of G and let $D_1 \subseteq V(G)$, if the induced subgraph $\langle V(G) - D_1\rangle$ is totally disconnected, D_1 is a minimal γ_{ss} set of G. Further A=$\{v_1, v_2, v_3, \ldots\ldots, v_n\}$⊂V(G) be the set of end vertices in G and $B = V(G) - A$. Then there exists vertex set $H \subseteq B$ such that $\forall\, v_j \in [V(G) - (H \cup A)]$ is adjacent to atleast one vertex of $H \cup A$ and $[V(G) - (H \cup A)]$. Then $\{H \cup A\}$ is a γ_r set of G. If $\langle H \cup A\rangle$ is regular, then (H $\cup A$) is a γ_{rr} set of G. Hence $|(H \cup A)| + |D_1| \leq |E(G)| + |D|$ which gives $\gamma_{rr}(G) + \gamma_{ss}(G)\leq q + \gamma_s(G)$.

A dominating set M is a strong dominating set, if for every vertex $u \in V - M$ there exist a vertex $v \in M$ with $deg\,(v) \geq deg\,(u)$ and u is adjacent to v. γ_{st}(G) is the minimum cardinality of a minimal strong dominating set, see[6].

Similarly, suppose a dominating set N is a weak dominating set, if for every vertex $u \in V - N$ and $v \in N$ with $deg\,(v) \leq deg\,(u)$ and u is adjacent to v. γ_w(G) is the minimum cardinality of minimal weak dominating set, see[5].

The following theorem relates $\gamma_w(G)$ and γ_{st}(G) with our concept.

Theorem 12: For any connected graph G, γ_{rr}(G)+$1 \leq \gamma_{st}$(G)+$\gamma_w(G)$ and $G \neq C_{n+3}$, $[n = 0,1,2,3 \ldots\ldots$except $G = C_{3n}$ and $G = C_{4n}$ with $n = 1,2,3, \ldots\ldots]$

Proof: Let V(G)= $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$ be the vertex set of G. Now for graph $G = C_{3n}$ and $G = C_{4n}$ with $n = 1,2,3, \ldots\ldots$, γ_{rr}(G)+$1 = 1 + 1 \leq \gamma_{st}$(G)+$\gamma_w(G) = 1 + 1$ and result holds to C_3. Also if $G = C_5$, $\gamma_{rr}(G)+1 = 5+1 > \gamma_{st}(G)+\gamma_w(G) = 2+2$. Hence $G \neq C_{n+3}$, $(n = 0,1,2,3 \ldots\ldots)$. Let $D=\{v_1,v_2,v_3, \ldots\ldots,v_k\} \subseteq V(G)$ be minimal set of vertices which covers all the vertices in G and suppose every $v_i \in V(G) - D$ is adjacent to atleast one vertex of D, D is a γ set of G. Suppose $V(G) - D = N$, $\forall v_j \in N$, $\deg(v_i) \geq \deg(v_j)$ $\forall v_i \in D$. Then $|D|=\gamma_{st}(G)$. Further suppose there exist a vertex set $D_1 \subseteq V(G)$ such that $\forall v_k \in D_1$ is adjacent to atleast one vertex of $V(G) - D_1$ and $N|D_1| = V(G)$, clearly D_1 is a dominating set of G. Further if $\forall v_i \in V(G) - D_1$, $\deg(v_m) \leq \deg(v_l)$ and $\forall v_m \in D_1$. Then D_1 is a weak dominating set. Suppose $A=\{v_1,v_2,v_3, \ldots\ldots,v_m\} \subset V(G)$ be the set of end vertices in G. Then $\{D \cup A\}$ forms a minimal restrained dominating set of G. Since each component of induced sub graph $|D \cup A|$ has same degree, then $(D \cup A)$ gives a γ_{rr} set of G. It follows that $|D \cup A| + 1 \leq |D| + |D_1|$ which gives $\gamma_{rr}(G)+1 \leq \gamma_{st}(G)+\gamma_w(G)$.

4. Conclusion

In this paper we surveyed selected results on Regular Restrained domination in graph. These results estabilish key relationship between the Regular restrained domination number and other parameters, including the domination number, the edge domination number, split domination number and entire domination number of a simple, and undirected graph.

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