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A Mathematical Model On Three Species Multi Ecology Consisting Of Host-Commensal-Mutualism

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Abstract

In this study, we develop a mathematical model to analyze the complex interactions among three species (S_1, S_2, S_3) in a multi-ecological system involving host-commensal-mutualistic relationships. The system comprises of two hosts (S_2, S_3) and two commensals (S_1, S_2) ie, S_2 is the host of S_1 and commensal of the host S_3. Further S_1 and S_3 are mutuals. Here all three species are having limited resources quantized by the respective carrying capacities. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths N_1, N_2, N_3 of S_1, S_2, S_3 respectively. All accessible critical points are recognized based on the primary model equations and criteria for their consistency are explained. If all the latent roots of the peculiar equation are either negative or zero then the model would be balanced otherwise imbalanced. Curvatures of the perturbations upon the critical points are analyzed. Further, we explain the universal consistency by appropriate Liapunov's method and the expansion rates of the species are numerically calculable victimization Runge-Kutta fourth order scheme.

Keywords: Balanced, commensal, critical point, host, imbalanced, latent root, mutualism, trajectory.

2010 Mathematics Subject Classification: 92D25; 92D40

INTRODUCTION

Ecology is the study of living organisms in relation to their environments, examining how they interact with their habitats and each other. As a branch of evolutionary biology, it seeks to understand the mechanisms regulating species in nature, including population dynamics, species distribution, and ecological relationships such as predator-prey interactions and competition.

The field of ecology is broadly categorized into autecology (the study of individual species populations) and synecology (the study of interactions among multiple species communities). Synecological research has led to the development of the ecosystem concept, which integrates living organisms-plants, animals, and microorganisms-with their physical surroundings. This foundational idea emerged from the collective work of generations of biologists, ecologists, and botanists. Theoretical ecology has been significantly advanced by researchers such as Gillman [3] and Kot [4], with contributions from both ecologists and mathematicians. Mathematical ecology, in particular, is divided into autecology and synecology, as explored in the works of Anna Sher [1], Arumugam [2], and Sharma [28].

The Role of Mathematical Modelling in Biological Systems

Mathematical models serve as vital tools in biological research, enabling scientists to analyze complex interactions through iterative data collection and theoretical simulations. When properly constructed, these models reveal relationships between physical variables and underlying processes, guiding experimental design and data interpretation. Given the complexity of real-life systems, mathematical formulations often replicate experimental outcomes without fully representing the actual mechanisms. Despite this, such models are invaluable in predicting system behavior and exploring interactions among different components. Empirical adjustments allow researchers to refine models, yielding insights applicable to real-world scenarios.

Several researchers have contributed to the field of biological modelling, Ma [6], Moghadas [7], Murray [8], and Sze-Bi Hsu [30] established foundational frame works. In Competitive Ecosystems: Srinivas [29] analyzed two- and three-species systems with limited/unlimited resources. Prey-Predator Dynamics: Narayan [9] studied models incorporating prey cover and alternate food sources for predators. Commensalism Models: Kumar [5] explored mathematical representations of commensal relationships. Syn-Ecosystems: Prasad [10–27] investigated continuous and discrete models for two-, three-, and four-species systems.

The current work presents an analytical and numerical examination of a three-species ecological system (S1, S2, S3) with limited resources. This study aims to deepen our understanding of such interactions through mathematical analysis and simulations.

MATHEMATICAL MODEL

Notation Appropriated

$N_i(t)$	:	The population strength of S_i at time t, $i=1,2,3$
t	:	Time instant
a_i	:	Natural growth rate of S_i, $i=1,2,3$
a_{ii}	:	Self inhibition coefficients of S_i, $i=1,2,3$
a_{13}, a_{23}	:	Interaction coefficients of S_1 due to S_3 and S_2 due to S_3
a_{12}, a_{31}	:	Interaction coefficients of S_1 due to S_2 and S_3 due to S_1
$k_i = \frac{a_i}{a_{ii}}$	:	Carrying capacities of S_i, $i=1,2,3$

Fundamental Equations

The model equations for syn ecology is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = N_1\left(a_1 - a_{11}N_1 + a_{12}N_2 + a_{13}N_3\right) \quad (1)$$

$$\frac{dN_2}{dt} = N_2\left(a_2 - a_{22}N_2 + a_{23}N_3\right) \quad (2)$$

$$\frac{dN_3}{dt} = N_3\left(a_3 - a_{33}N_3 + a_{31}N_1\right) \quad (3)$$

CRITICAL POINTS

The system under investigation has eight critical points $\frac{dN_i}{dt} = 0, i = 1,2,3$ at given by

Fully washed out state.

$E_1: \ \overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0$

States in which only two of the tree species are washed out while the other one is not.

$E_2: \ \overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=k_3$

$E_3: \ \overline{N_1}=0, \overline{N_2}=k_2, \overline{N_3}=0$

$E_4: \overline{N_1}=k_1, \ \overline{N_2}=0, \overline{N_3}=0$

States in which only one of the tree species is washed out while the other two are not.

$E_5: \overline{N_1}=0, \overline{N_2}=k_2+\frac{a_{23}k_3}{a_{22}}, \overline{N_3}=k_3$

$E_6: \overline{N_1}=k_1+\frac{a_{13}k_3}{a_{11}}, \overline{N_2}=0, \overline{N_3}=k_3$

$E_7: \ \overline{N_1}=k_1+\frac{a_{12}k_2}{a_{11}}, \overline{N_2}=k_2, \overline{N_3}=0$

The normal steady state.

$$E_8: \overline{N}_1=\frac{a_1a_{22}a_{33}+a_{12}a_2a_{33}+a_{12}a_{23}a_3+a_{13}a_{22}a_3}{a_{11}a_{22}a_{33}-\left(a_{12}a_{23}a_{31}+a_{13}a_{22}a_{31}\right)};$$

$$\overline{N}_2=\frac{a_{11}a_2a_{33}+a_{11}a_{23}a_3+a_1a_{23}a_{31}-a_{13}a_2a_{31}}{a_{11}a_{22}a_{33}-\left(a_{12}a_{23}a_{31}+a_{13}a_{22}a_{31}\right)};$$

$$\overline{N}_3=\frac{a_{11}a_{22}a_3+a_{12}a_2a_{31}+a_1a_{22}a_{31}}{a_{11}a_{22}a_{33}-\left(a_{12}a_{23}a_{31}+a_{13}a_{22}a_{31}\right)}$$

CONSISTENCY ANALYSIS

Let $N=\left(N_1,N_2,N_3\right)=\overline{N}+U$

where $U=\left(u_1,u_2,u_3\right)^T$ is very small perturbation upon the critical point $\overline{N}=\left(\overline{N}_1,\overline{N}_2,\overline{N}_3\right)$.

The fundamental equations (1), (2) and (3) are quasi-linearized to obtain the equations for the perturbed state as, $\frac{dU}{dt}=AU$

Where

$$A(E)=\begin{bmatrix} a_1-2a_{11}\overline{N}_1+a_{12}\overline{N}_2+a_{13}\overline{N}_3 & a_{12}\overline{N}_1 & a_{13}\overline{N}_1 \\ 0 & a_2-2a_{22}\overline{N}_2+a_{23}\overline{N}_3 & a_{23}\overline{N}_2 \\ a_{31}\overline{N}_3 & 0 & a_3-2a_{33}\overline{N}_3+a_{31}\overline{N}_1 \end{bmatrix} \qquad (4)$$

The peculiar equation for the scheme is $|A-\lambda I|=0$

If all the latent roots are either negative or zero, then critical point is balanced otherwise imbalanced.

Consistency of $E_1: \ \overline{N_1}=0, \overline{N_2}=0, \overline{N_3}=0$

In this state, we have $A\left(E_1\right)=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

The peculiar equation is given by $\left(\lambda-a_1\right)\left(\lambda-a_2\right)\left(\lambda-a_3\right)=0$

The latent roots are a_1,a_2,a_3 .Since all the three roots are positive.

Hence, the point is imbalanced and the solutions are

$$u_1 = u_{10}e^{a_1 t}\,;\, u_2 = u_{20}e^{a_2 t}\,;\; u_3 = u_{30}e^{a_3 t} \qquad (5)$$

Where u_{10}, u_{20}, u_{30} are the primary values of u_1, u_2, u_3 respectively.

The trajectories in $u_1 - u_2$, $u_2 - u_3$ and $u_3 - u_1$ planes are $(x_1)^{\frac{1}{a_1}} = (x_2)^{\frac{1}{a_2}} = (x_3)^{\frac{1}{a_3}}$

Where $x_1 = \frac{u_1}{u_{10}}, x_2 = \frac{u_2}{u_{20}}$ and $x_3 = \frac{u_3}{u_{30}}$

Consistency of E_2 : $\overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = k_3$

At this critical point, we have $A(E_2) = \begin{bmatrix} a_1 + a_{13}k_3 & 0 & 0 \\ 0 & a_2 + a_{23}k_3 & 0 \\ a_{31}k_3 & 0 & -a_3 \end{bmatrix}$

$a_1 + a_{13}k_3, a_2 + a_{23}k_3, -a_3$ are the latent roots. Here only one of these three roots is negative .Hence the point is imbalanced, the solution curves are

$$u_1 = u_{10}e^{(a_1 + a_{13}k_3)t}\,;\, u_2 = u_{20}e^{(a_2 + a_{23}k_3)t}\,;\, u_3 = u_{30}e^{-a_3 t} \qquad (6)$$

And the trajectories of perturbations are given by $(x_1)^{\frac{1}{a_1 + a_{13}k_3}} = (x_2)^{\frac{1}{a_2 + a_{23}k_3}} = (x_3)^{\frac{-1}{a_3}}$

Consistency of E_3 : $\overline{N_1} = 0, \overline{N_2} = k_2, \overline{N_3} = 0$

In this case, we have $A(E_3) = \begin{bmatrix} a_1 + a_{12}k_2 & 0 & 0 \\ 0 & -a_2 & a_{23}k_2 \\ 0 & 0 & a_3 \end{bmatrix}$

The latent roots are $a_1 + a_{12}k_2, -a_2, a_3$ **.** Since one of the three roots is lesser than zero and other two roots are greater than zero, hence the point is **imbalanced.**

The solutions are

$$u_1 = u_{10}e^{(a_1 + a_{12}k_2)t}\,;\, u_2 = \left[u_{20} - \frac{a_{23}k_2u_{30}}{a_2 + a_3}\right]e^{-a_2 t} + \frac{a_{23}k_2u_{30}}{a_2 + a_3}e^{a_3 t} \text{ and } u_3 = u_{30}e^{a_3 t} \qquad (7)$$

The curvatures of the perturbations of the above equations are

$$(x_1)^{\frac{1}{a_1 + a_{12}k_2}} = (x_3)^{\frac{1}{a_3}}\,;\; x_2 = \left[1 - \frac{a_{23}k_2u_{30}}{(a_2 + a_3)u_{20}}\right](x_1)^{\frac{-a_2}{a_1 + a_{12}k_2}} + \frac{a_{23}k_2u_{30}}{(a_2 + a_3)u_{20}}(x_1)^{\frac{a_3}{a_1 + a_{12}k_2}}$$

$$x_2 = \left[1 - \frac{a_{23}k_2u_{30}}{(a_2 + a_3)u_{20}}\right](x_3)^{\frac{-a_2}{a_3}} + \frac{a_{23}k_2u_{30}x_3}{(a_2 + a_3)u_{20}}$$

Consistency of E_4 : $\overline{N_1} = k_1, \overline{N_2} = 0, \overline{N_3} = 0$

At this critical point the matrix given by $A(E_4) = \begin{bmatrix} -a_1 & a_{12}k_3 & a_{13}k_1 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 + a_{31}k_1 \end{bmatrix}$

The latent roots are $-a_1, a_2, a_3 + a_{31}k_1$ .Since one the three roots is lesser than zero and other two roots are greater than zero, hence the point is imbalanced. The equations yield the solutions,

$$u_1 = (u_{10} - \beta_2 - \beta_3)e^{-a_1 t} + \beta_2 e^{a_2 t} + \beta_3 e^{(a_3 + a_{31}k_1)t}\,;\, u_2 = u_{20}e^{a_2 t}\,;\, u_3 = u_{30}e^{(a_3 + a_{31}k_1)t} \qquad (8)$$

Where $\beta_2 = \frac{a_{12}k_1u_{20}}{a_2 + a_1} > 0$ and $\beta_3 = \frac{a_{13}k_1u_{30}}{a_1 + a_3 + a_{31}k_1} > 0$

The curvatures of the perturbations of perturbed species are given by $(x_2)^{\frac{1}{a_2}} = (x_3)^{\frac{1}{a_3+a_{31}k_1}}$ and

$$x_1 = \left[1 - \frac{\beta_2 + \beta_3}{u_{10}}\right](x_2)^{\frac{-a_1}{a_2}} + \frac{\beta_2 x_2}{u_{10}} + \frac{\beta_3}{u_{10}}(x_2)^{\frac{a_3}{a_2}}\,;\; x_1 = \left[1 - \frac{\beta_2 + \beta_3}{u_{10}}\right](x_3)^{\frac{-a_1}{a_3}} + \frac{\beta_2}{u_{10}}(x_3)^{\frac{a_2}{a_3}} + \frac{\beta_3 x_3}{u_{10}}$$

Consistency of $E_5 : \overline{N_1} = 0, \overline{N_2} = k_2 + \frac{a_{23}k_3}{a_{22}}, \overline{N_3} = k_3$

In this case, the matrix $A(E_5)$ is given by $A(E_5) = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & -(a_2 + a_{23}k_2) & \gamma_2 \\ a_{31}k_3 & 0 & -a_3 \end{bmatrix}$

Where $\gamma_1 = \left(a_1 + a_{12}k_2 + \frac{a_{12}a_{23}k_3}{a_{22}} + a_{13}k_3\right) > 0$ and $\gamma_2 = \left(a_{23}k_2 + \frac{a_{23}^2 k_3}{a_{22}}\right) > 0$

The latent roots of E_5 are $\gamma_1, -(a_2 + a_{23}k_2), -a_3$. Since one of the three roots is greater than zero and other two roots are lesser than zero. Hence, the point E_5 is imbalanced. The solution curves are given by

$$u_1 = u_{10}e^{\gamma_1 t}\,; u_2 = \left[u_{20} - \frac{\gamma_2 u_{30}}{(a_2 - a_3 + a_{23}k_2)}\right]e^{-(a_2+a_{23}k_2)t} + \frac{\gamma_2 u_{30}}{(a_2 - a_3 + a_{23}k_2)}e^{-a_3 t}\,; u_3 = u_{30}e^{-a_3 t} \quad (9)$$

The curvatures of the perturbations of perturbed species are given by $(x_1)^{\frac{1}{\gamma_1}} = (x_3)^{-\frac{1}{a_3}}$ and

$$x_2 = \left[1 - \frac{\gamma_2 u_{30}}{(a_2 - a_3 + a_{23}k_2)u_{20}}\right](x_1)^{\frac{-(a_2+a_{23}k_2)}{\gamma_1}} + \frac{\gamma_2 u_{30}}{(a_2 - a_3 + a_{23}k_2)u_{20}}(x_1)^{\frac{-a_3}{\gamma_1}}$$

$$x_2 = \left[1 - \frac{\gamma_2 u_{30}}{(a_2 - a_3 + a_{23}k_2)u_{20}}\right](x_3)^{\frac{a_2+a_{23}k_2}{a_3}} + \frac{\gamma_2 u_{30} x_3}{(a_2 - a_3 + a_{23}k_2)u_{20}}$$

Consistency of $E_6 : \overline{N_1} = k_1 + \frac{a_{13}k_3}{a_{11}}, \overline{N_2} = 0, \overline{N_3} = k_3$

At this point, we have $A(E_6) = \begin{bmatrix} -(a_1 + a_{13}k_3) & \delta_2 & \delta_3 \\ 0 & a_2 + a_{23}k_3 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}$

Where $\delta_2 = \left(a_{12}k_1 + \frac{a_{12}a_{13}}{a_{11}}k_3\right) > 0, \delta_3 = \left(a_{13}k_1 + \frac{a_{13}^2}{a_{11}}k_3\right) > 0$

Here, the latent roots are $-(a_1 + a_{13}k_3), a_2 + a_{23}k_3$ and $-a_3$. Since two of the three roots $-(a_1 + a_{13}k_3), -a_3$ are lesser than zero and one root $a_2 + a_{23}k_3$ is greater than zero. Hence the point E_6 is imbalanced and the solution curves are

$$u_1 = [u_{10} - B_1 - B_2]e^{-(a_1+a_{13}k_3)t} + B_1 e^{(a_2+a_{23}k_3)t} + B_2 e^{-a_3 t}\,;\; u_2 = u_{20}e^{(a_2+a_{23}k_3)t}\,; u_3 = u_{30}e^{-a_3 t} \quad (10)$$

Where $B_1 = \frac{\delta_2 u_{20}}{a_1 + a_2 + a_{13}k_3 + a_{23}k_3} > 0; B_2 = \frac{\delta_3 u_{30}}{(a_1 + a_{13}k_3) - a_3}$ with $(a_1 + a_{13}k_3) \neq a_3$

The curvatures of the perturbations are given by

$$(x_2)^{\frac{1}{a_2+a_{23}k_3}} = (x_3)^{-\frac{1}{a_3}};\ x_1 = \left[\frac{u_{10}-B_1-B_2}{u_{10}}\right](x_2)^{-\frac{a_1+a_{13}k_3}{a_2+a_{23}k_3}} + \frac{B_1x_2}{u_{10}} + \frac{B_2}{u_{10}}(x_2)^{-\frac{a_3}{a_2+a_{23}k_3}} \text{ and}$$

$$x_1 = \left[\frac{u_{10}-B_1-B_2}{u_{10}}\right](x_3)^{-\frac{a_1+a_{13}k_3}{a_3}} + \frac{B_1x_3}{u_{10}}(x_3)^{-\frac{a_2+a_{23}k_3}{a_3}} + \frac{B_2x_3}{u_{10}}$$

Consistency of E_7: $\overline{N_1} = k_1 + \frac{a_{12}k_2}{a_{11}}, \overline{N_2} = k_2, \overline{N_3} = 0$

At this critical point, the matrix $A(E_7)$ is given by $A(E_7) = \begin{bmatrix} -(a_1+a_{12}k_2) & \mu_2 & \mu_3 \\ 0 & -a_2 & a_{23}k_2 \\ 0 & 0 & \mu_4 \end{bmatrix}$

Where $\mu_2 = a_{12}k_1 + \frac{a_{12}^2k_2}{a_{11}} > 0$, $\mu_3 = a_{13}\left(k_1 + \frac{a_{12}k_2}{a_{11}}\right) > 0$, $\mu_4 = \left(a_3 + a_{31}k_1 + \frac{a_{31}a_{12}k_2}{a_{11}}\right) > 0$

The latent roots are $-(a_1+a_{12}k_2), -a_2, \mu_4$. Since one of the three roots μ_4 is greater than zero, hence the point E_7 is imbalanced.

The solution curves are given by

$$u_1 = \left[u_{10} - (X_1 + X_2)\right]e^{-(a_1+a_{12}k_2)t} + X_1e^{-a_2t} + X_2e^{\mu_4t}$$

$$u_2 = (u_{20} - \rho_1)e^{-a_2t} + \rho_1e^{\mu_4t};\ u_3 = u_{30}e^{\mu_4t} \qquad (11)$$

where $X_1 = \frac{\mu_2(u_{20}-\rho_1)}{(a_1+a_{12}k_2)-a_2}$, with $(a_1+a_{12}k_2) \neq a_2; X_2 = \frac{\mu_2\rho_1 + \mu_3u_{30}}{a_1+a_{12}k_2+\mu_4} > 0; \rho_1 = \frac{a_{23}k_2u_{30}}{a_2+a_3} > 0$ and

$x_1 = \left(\frac{u_{10}-X_1-X_2}{u_{10}}\right)x_3^{\frac{-(a_1+a_{12}k_2)}{\mu_4}} + \frac{X_1}{u_{10}}x_3^{\frac{-a_2}{\mu_4}} + \frac{X_2x_3}{u_{10}};\ x_2 = \left(\frac{u_{20}-\rho_1}{u_{20}}\right)x_3^{\frac{-a_2}{\mu_4}} + \frac{\rho_1x_3}{u_{20}}$ are the trajectories of perturbed species.

Consistency of E_8: $\overline{N}_1 = \frac{a_1a_{22}a_{33} + a_{12}a_2a_{33} + a_{12}a_{23}a_3 + a_{13}a_{22}a_3}{a_{11}a_{22}a_{33} - (a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31})}$;

$$\overline{N}_2 = \frac{a_{11}a_2a_{33} + a_{11}a_{23}a_3 + a_1a_{23}a_{31} - a_{13}a_2a_{31}}{a_{11}a_{22}a_{33} - (a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31})};$$

$$\overline{N}_3 = \frac{a_{11}a_{22}a_3 + a_{12}a_2a_{31} + a_1a_{22}a_{31}}{a_{11}a_{22}a_{33} - (a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31})}$$

In this critical point, we get $A(E_8) = \begin{bmatrix} a_1-\sigma_1 & a_{12}\alpha_8 & a_{13}\alpha_8 \\ 0 & a_2-\sigma_2 & a_{23}\beta_8 \\ a_{31}\gamma_8 & 0 & a_3-\sigma_3 \end{bmatrix}$

Where $\sigma_1 = \left(a_1 - 2a_{11}\overline{N}_1 + a_{12}\overline{N}_2 + a_{13}\overline{N}_3\right)$

$\sigma_2 = \left(a_2 - 2a_{22}\overline{N}_2 + a_{23}\overline{N}_3\right)$ and $\sigma_3 = \left(a_3 - 2a_{33}\overline{N}_3 + a_{31}\overline{N}_1\right)$

After applying $|A(E_8) - \lambda I| = 0$,

the characteristic equation will be $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ (12)

Where $b_1 = a_{11}\overline{N_1} + a_{22}\overline{N_2} + a_{33}\overline{N_3}$

$$b_2 = a_{22}a_{33}\overline{N_2}\,\overline{N_3} + \left(a_{11}a_{22} - a_{12}a_{21}\right)\overline{N_1}\,\overline{N_2} + \left(a_{11}a_{33} - a_{13}a_{31}\right)\overline{N_1}\,\overline{N_3}$$
$$b_3 = \left(a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}\right)\overline{N_1}\,\overline{N_2}\,\overline{N_3}$$

According to Routh-Hurwitz's criteria, the necessary and sufficient conditions for local stability of co-existent points are $b_1 > 0, b_3 > 0 \,\&\, b_3\left(b_1b_2 - b_3\right) > 0$

It is evident that $b_1 > 0$ and $a_{11}a_{22}a_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3} > \left(a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}\right)\overline{N_1}\,\overline{N_2}\,\overline{N_3}$

Thus the stability of co-existent state is determined by the sign of $b_1b_2 - b_3$.

By the calculations, we obtain

$$b_1b_2 - b_3 = \left(a_{11}^2a_{22} - a_{11}a_{12}a_{21}\right)\overline{N_1}^2\,\overline{N_2} + \left(a_{11}^2a_{33} - a_{11}a_{13}a_{31}\right)\overline{N_1}^2\,\overline{N_3} + a_{22}^2a_{33}\overline{N_2}^2\,\overline{N_3}$$
$$+\left(a_{22}^2a_{11} - a_{12}a_{21}a_{22}\right)\overline{N_2}^2\,\overline{N_1} + a_{33}^2a_{22}\overline{N_3}^2\,\overline{N_2} + \left(a_{33}^2a_{11} - a_{13}a_{31}a_{33}\right)\overline{N_3}^2\,\overline{N_1} + 2a_{11}a_{22}a_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3} > 0$$

Hence the co-existent state is locally asymptotically stable.

Let $\lambda_1, \lambda_2, \lambda_3$ be the three roots of the equation (12), then the solution of the perturbation equations is

$$u_1 = u_{10}\left(A_1e^{\lambda_1 t} + B_1e^{\lambda_2 t} + C_1e^{\lambda_3 t}\right)$$
$$u_2 = u_{20}\left(A_2e^{\lambda_1 t} + B_2e^{\lambda_2 t} + C_2e^{\lambda_3 t}\right) \text{ and}$$
$$u_3 = u_{30}\left(A_3e^{\lambda_1 t} + B_3e^{\lambda_2 t} + C_3e^{\lambda_3 t}\right) \qquad (13)$$

Where $A_1 = u_{10}\left[\lambda_1^2 + \lambda_1E_1 + F_1\right] + G_1u_{20} + H_1u_{30}$

$E_1 = a_{22}\overline{N}_2 + a_{33}\overline{N}_3, F_1 = a_{22}a_{33}\overline{N}_2\overline{N}_3, G_1 = a_{13}a_{32}\overline{N}_2\overline{N}_1$ and $H_1 = a_{13}a_{22}\overline{N_1}\,\overline{N_2} + \lambda_1a_{13}\overline{N_1}$

$$B_1 = \frac{u_{10}\left[\lambda_2^2 + \lambda_2E_2 + F_2\right] + G_2u_{20} + H_2u_{30}}{\left(\lambda_2 - \lambda_1\right)\left(\lambda_2 - \lambda_3\right)}$$

$E_2 = a_{22}\overline{N}_2 + a_{33}\overline{N}_3, F_2 = a_{22}a_{23}\overline{N}_1\overline{N}_3 + a_{33}\overline{N}_3, G_2 = a_{12}\overline{N}_1\lambda_3 - a_{12}\overline{N}_1 - a_{12}\lambda_2\overline{N}_1$

$H_2 = a_{13}\overline{N}_1\lambda_3 - a_{13}\overline{N}_1 - a_{12}a_{23}\overline{N}_1\overline{N}_2 - a_{13}a_{22}\overline{N}_1\overline{N}_2 - a_{13}\overline{N}_1\lambda_2$

$$C_1 = \frac{u_{10}\left[\lambda_3^2 + \lambda_3E_3 + F_3\right] + G_3u_{20} + H_3u_{30}}{\left(\lambda_3 - \lambda_1\right)\left(\lambda_3 - \lambda_2\right)}$$

$E_3 = a_{22}\overline{N}_2 + a_{33}\overline{N}_3, F_3 = a_{22}a_{23}\overline{N}_1\overline{N}_3, G_3 = a_{12}\overline{N}_1$ and

$H_3 = a_{13}\overline{N}_1 + a_{12}a_{23}\overline{N}_1\overline{N}_2 + a_{13}a_{22}\overline{N}_1\overline{N}_2 + a_{12}a_{23}\overline{N}_1\overline{N}_3$

$$A_2 = \frac{u_{20}\left[\lambda_1^2 + \lambda_1P_1 + Q_1\right] + R_1u_{10} + T_1u_{30}}{\left(\lambda_1 - \lambda_2\right)\left(\lambda_1 - \lambda_3\right)}$$

$P_1 = a_{11}\overline{N}_1 + a_{33}\overline{N}_3, Q_1 = \left(a_{11}a_{33} - a_{13}a_{31}\right)\overline{N}_1\overline{N}_3, R_1 = a_{21}\overline{N}_2\lambda_1, T_2 = a_{23}\overline{N}_2$

$$B_2 = \frac{u_{20}\left[\lambda_2^2 + \lambda_2P_2 + Q_2\right] + R_2u_{10} + T_2u_{30}}{\left(\lambda_2 - \lambda_1\right)\left(\lambda_2 - \lambda_3\right)}$$

$P_2 = a_{23}\overline{N}_3 + a_{11}\overline{N}_1, Q_2 = a_{31}a_{13}\overline{N}_1\overline{N}_3, R_2 = a_{31}a_{23}\overline{N}_2\overline{N}_3, T_2 = a_{23}\overline{N}_2\lambda_3 + a_{13}a_{23}\overline{N}_1\overline{N}_2$

$$C_2 = \frac{u_{10}\left[\lambda_3^2 + \lambda_3P_3 + Q_3\right] + R_3u_{10} + T_3u_{30}}{\left(\lambda_3 - \lambda_1\right)\left(\lambda_3 - \lambda_2\right)}$$

$P_3 = a_{23}\overline{N}_3 + a_{11}\overline{N}_1, Q_3 = a_{31}a_{13}\overline{N}_1\overline{N}_3, R_3 = a_{31}a_{23}\overline{N}_2\overline{N}_3, T_3 = a_{23}\overline{N}_2\lambda_3 + a_{11}a_{23}\overline{N}_1\overline{N}_2$

$$A_3 = \frac{u_{30}\left[\lambda_1^2 + \lambda_1P_1' + Q_1'\right] - R_1'u_{20} + T_1'u_{10}}{\left(\lambda_1 - \lambda_2\right)\left(\lambda_1 - \lambda_3\right)}$$

$$P_1' = a_{11}\overline{N}_1 + a_{22}N_2, Q_1' = a_{11}a_{22}\overline{N}_1\overline{N}_2, R_1' = a_{31}a_{21}\overline{N}_1, T_1' = a_{31}\lambda_3 + a_{22}a_{31}\overline{N}_2$$

$$B_3 = \frac{u_{30}\left[\lambda_2^2 + \lambda_2 P_1' + Q_1'\right] - R_1' u_{20} + T_1' u_{10}}{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}$$

$$P_2' = a_{11}\overline{N}_1 + a_{22}\overline{N}_2, Q_2' = a_{11}a_{22}\overline{N}_1\overline{N}_2, R_2' = a_{31}a_{21}\overline{N}_1, T_1' = a_{31}\lambda_3 + a_{22}a_{31}\overline{N}_2$$

$$C_3 = \frac{u_{30}\left[\lambda_3^2 + \lambda_3 P'_3 + Q'_3\right] + R'_3 u_{20} + T_3' u_{10}}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}$$

$$P'_3 = a_{22}\overline{N}_2 + a_{11}\overline{N}_1, Q'_3 = a_{11}a_{22}\overline{N}_1\overline{N}_2, R'_3 = a_{31}a_{21}\overline{N}_1, T'_3 = a_{31}\lambda_3 + a_{22}a_{31}\overline{N}_2$$

LIAPUNOV'S METHOD OF UNIVERSAL CONSISTENCY:

We discussed the local consistency of all eight critical points. From which only one point $E_8\left(\overline{N}_1, \overline{N}_2, \overline{N}_3\right)$ is balanced and rest of them are imbalanced. We now examine the universal consistency of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

Theorem : The equilibrium state $E_8\left(\overline{N}_1, \overline{N}_2, \overline{N}_3\right)$ is globally asymptotically stable.

Proof Let us consider the following Liapunov's function

$$V(N_1, N_2, N_3) = N_1 - \overline{N}_1 - \overline{N}_1 \log\left(\frac{N_1}{\overline{N}_1}\right) + d_1\left[N_2 - \overline{N}_2 - \overline{N}_2 \log\left(\frac{N_2}{\overline{N}_2}\right)\right] + d_2\left[N_3 - \overline{N}_3 - \overline{N}_3 \log\left(\frac{N_3}{\overline{N}_3}\right)\right]$$

where d_1 and d_2 are suitable constants to be determined as in the subsequent steps.

Now, the time derivative of V, along with solutions of (1), (2) and (3) can be written as

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\frac{dN_1}{dt} + d_1\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\frac{dN_2}{dt} + d_2\left(\frac{N_3 - \overline{N}_3}{N_3}\right)\frac{dN_3}{dt}$$

$$\frac{dV}{dt} = \left(\frac{N_1 - \overline{N}_1}{N_1}\right)\left(a_1 N_1 - a_{11}N_1^2 + a_{12}N_1N_2 + a_{13}N_1N_3\right)$$

$$+d_1\left(\frac{N_2 - \overline{N}_2}{N_2}\right)\left(a_2 N_2 - a_{22}N_2^2 + a_{23}N_2N_3\right) + d_2\left(\frac{N_3 - \overline{N}_3}{N_3}\right)\left(a_3 N_3 - a_{33}N_3^2 + a_{31}N_1N_3\right)$$

$$= -a_{11}\left(N_1 - \overline{N}_1\right)^2 + a_{12}\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right) + a_{13}\left(N_1 - \overline{N}_1\right)\left(N_3 - \overline{N}_3\right)$$

$$+d_1\left[-a_{22}\left(N_2 - \overline{N}_2\right)^2 + a_{23}\left(N_3 - \overline{N}_3\right)\left(N_2 - \overline{N}_2\right)\right] + d_2\left[a_{33}\left(N_3 - \overline{N}_3\right)^2 + a_{31}\left(N_3 - \overline{N}_3\right)\left(N_1 - \overline{N}_1\right)\right]$$

$$= -\left[\sqrt{a_{11}}\left(N_1 - \overline{N}_1\right) + \sqrt{d_1 a_{22}}\left(N_2 - \overline{N}_2\right) + \sqrt{d_2 a_{33}}\left(N_3 - \overline{N}_3\right)\right]^2 + \left(2\sqrt{d_1 a_{11} a_{12}} + a_{12}\right)\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right)$$

$$+\left(2\sqrt{d_2 a_{11} a_{33}} + a_{13} + d_2 a_{31}\right)\left(N_1 - \overline{N}_1\right)\left(N_3 - \overline{N}_3\right) + \left(2\sqrt{d_1 d_2 a_{22} a_{33}} + d_1 a_{22}\right)\left(N_2 - \overline{N}_2\right)\left(N_3 - \overline{N}_3\right) \quad (14)$$

The positive constants l_1 and l_2 as so chosen that, the coefficients of $\left(N_1 - \overline{N}_1\right)\left(N_2 - \overline{N}_2\right)$, $\left(N_1 - \overline{N}_1\right)\left(N_3 - \overline{N}_3\right)$ and $\left(N_2 - \overline{N}_2\right)\left(N_3 - \overline{N}_3\right)$ in (14) vanish.

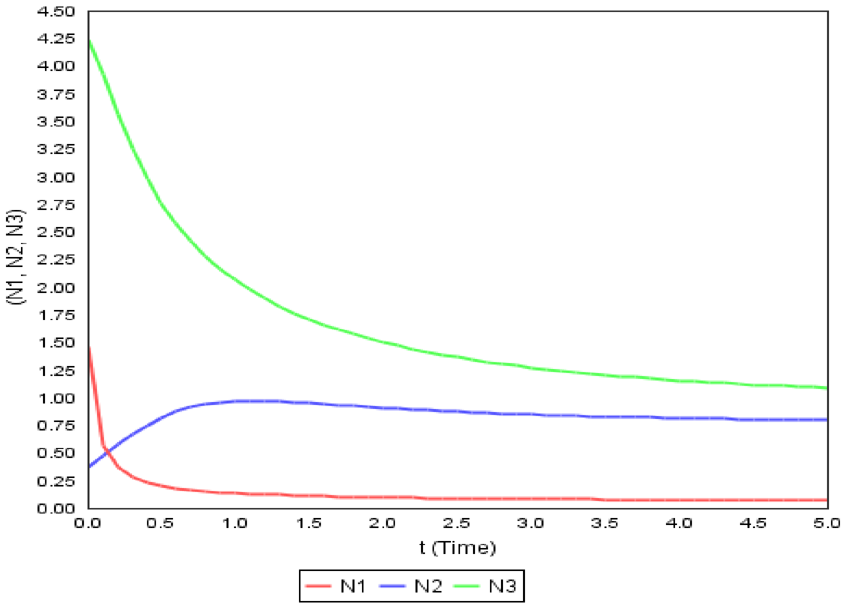
Then we have $d_1 = \frac{a_{12}^2}{4(a_{11}a_{22})} > 0$ and $d_2 = \frac{a_{12}^2 a_{23}^2}{16 a_{11} a_{22}^2 a_{33}} > 0$, with this choice of the constants d_1 and d_2.

$$\frac{dV}{dt} = -\sqrt{a_{11}}\left[\left(N_1 - \overline{N}_1\right) + \frac{a_{12}}{2a_{11}}\left(N_2 - \overline{N}_2\right) + \frac{a_{12}a_{23}}{4a_{11}a_{22}}\left(N_3 - \overline{N}_3\right)\right]^2 < 0$$

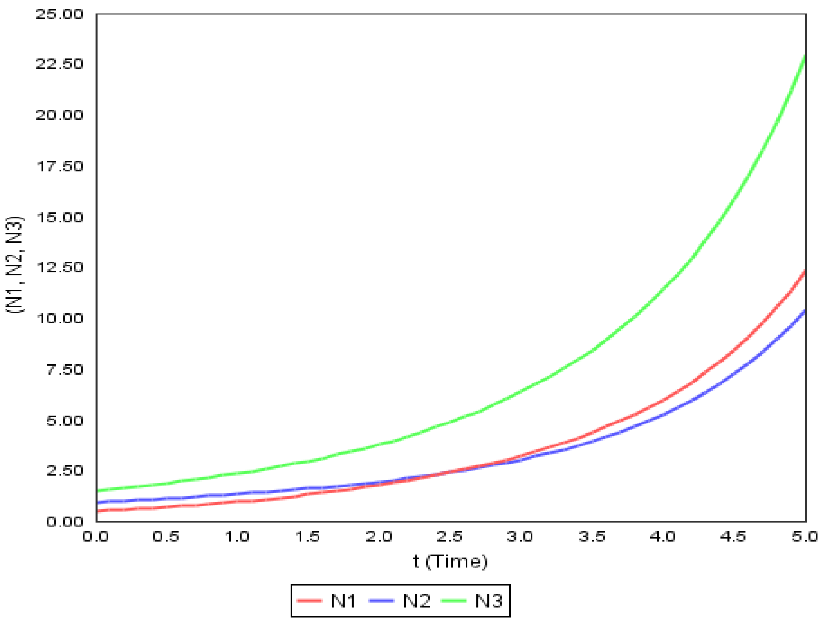
Hence, the steady state is universally asymptotical balanced.

NUMERICAL APPROACH

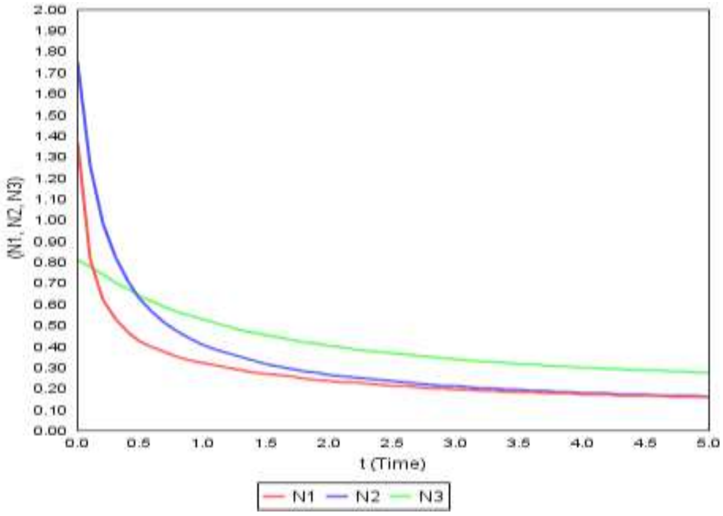
The numerical solutions of the growth rate equations computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures from1 to 8.



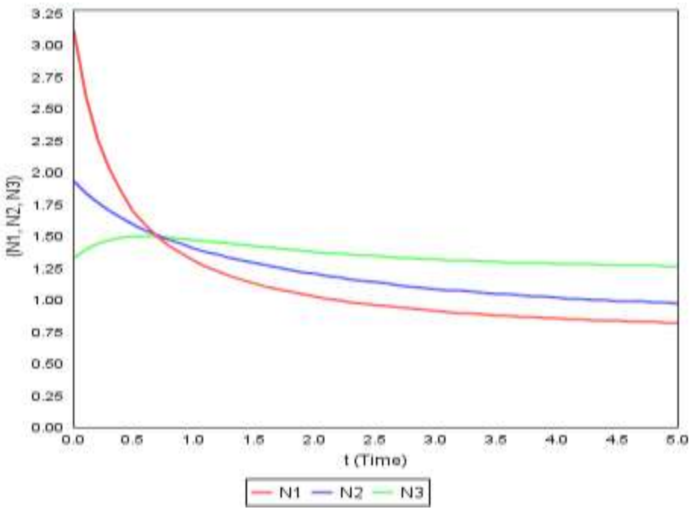
Figure(1): Variation of N_1, N_2 and N_3 against time (t) for a1=0.234;a11=12.96;a12=0.414;a13=0.306;a2=1.314;a22=2.34;’a23=0.486;a3=0.414;a33=0.486;a31=0.954, N1[t0]=1.458;N2[t0]=0.378;N3[t0]=4.23



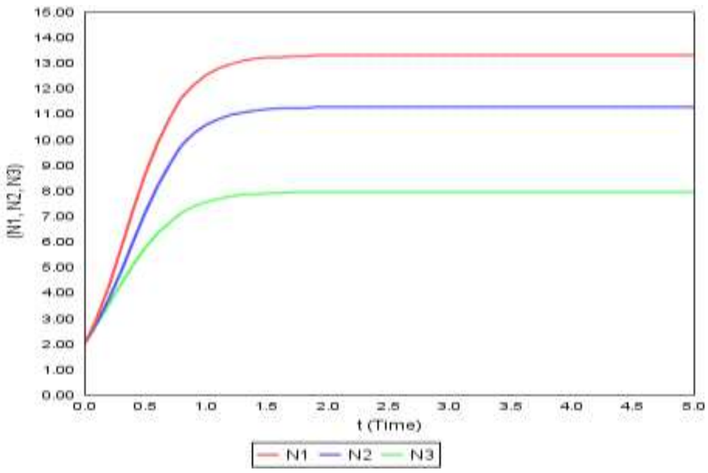
Figure(2) : Variation of N_1, N_2 and N_3 against time (t) for a1=0.07;a11=1.488,a12=0.16;a13=0.76;a2=0.552;a22=0.84;a23=0.392; a3=0.6;a33=0.288;a31=0.552,N1[t0]=0.504;N2[t0]=0.95;N3[t0]=1.512



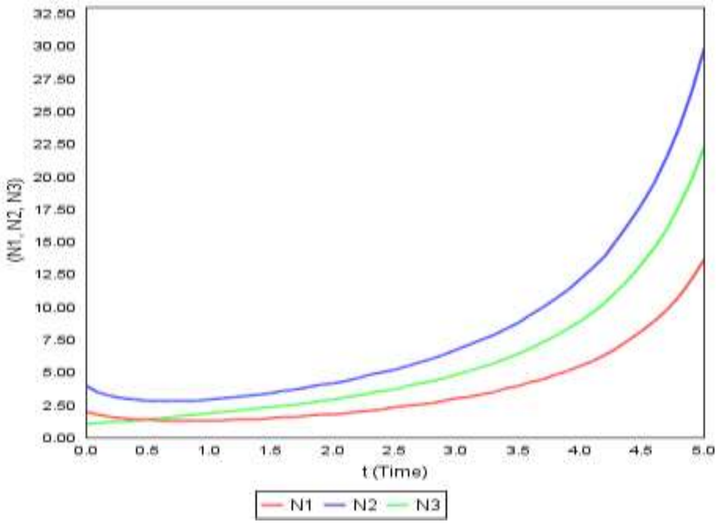
Figure(3) : Variation of N_1, N_2 and N_3 against time (t) for
a1=0.13;a11=9.1;a12=0.4;a13=4.2;a2=0.2;a22=2.7;a23=0.53;
a3=0.23;a33=1.33;a31=0.4;N1[t0]=1.37;N2[t0]=1.75;N3[t0]=0.81



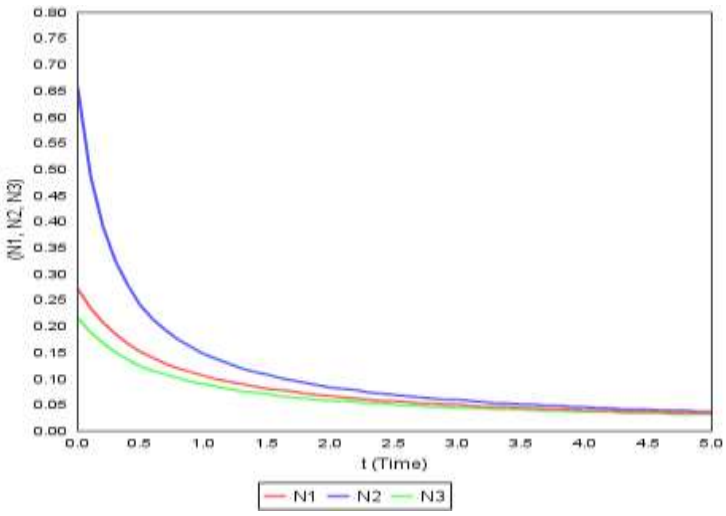
Figure(4) : Variation of N_1, N_2 and N_3 against time (t) for
a1=0.17;a11=0.97;a12=0.13;a13=0.37;a2=0.1;a22=0.53;a23=0.3;
a3=0.97; a33=0.97;a31=0.3;N1[t0]=3.12;N2[t0]=1.93;N3[t0]=1.33



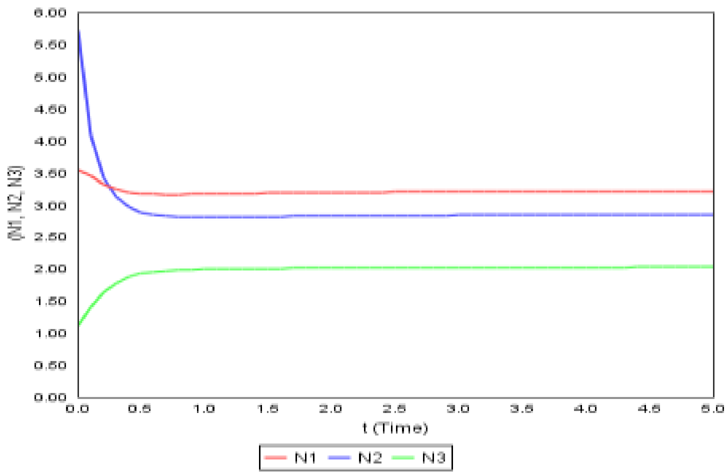
Figure(5) : Variation of N_1, N_2 and N_3 against time (t) for
a1=3.872; a11=1.232;a12=0.704;a13=0.576; a2=3.736;
a22=0.552;a23=0.312, a3=4.736;a33=0.944;a31=0.208;N1[t0]=2.0;N2[t0]=2.0;N3[t0]=2.0



Figure(6) : Variation of N_1, N_2 and N_3 against time (t) for a1=0.16;a11=2.696; a12=0.84;a13=0.576;a2=0.576; a22=0.84;a23=1.152; a3=0.336;a33=0.184;a31=0.368;N1[t0]=2.0;N2[t0]=4.0;N3[t0]=1.0



Figure(7) : Variation of N_1, N_2 and N_3 against time (t) for a1=0.056;a11=6.952;a12=0.312;a13=0.288;a2=0.001;a22=5.336;a23=0.184; a3=0.104;a33=7.608;a31=0.128;N1[t0]=0.272;N2[t0]=0.656;N3[t0]=0.216



Figure(8) : Variation of N_1, N_2 and N_3 against time (t) for a1=0.784;a11=1.488;a12=0.552;a13=1.2;a2=0.704;a22=1.232;a23=1.384; a3=0.576;a33=2.48;a31=1.384;N1[t0]=3.544;N2[t0]=5.72;N3[t0]=1.128

OBSERVATIONS OF THE GRAPHS

Case 1: In this case initially the first species dominates over the second species till the time instant $t^* = 0.13$ after which the dominance is reversed. The natural growth rate of the second species is greater than the natural growth rates of both first and third species. It is evident that the third species dominates at the initial time point, with a significantly higher population compared to first and second. Further, the self inhibition coefficient of the first species is highest. This is shown in Figure 1.

Case 2: In this situation the second species dominates over the first species up to the time instant $t^* = 2.65$ after which the dominance is reversed. The initial values of the first, second and third species are in ascending order. The first species with low natural birth rate. Further, it is evident that the birth rate of the second species and the interaction coefficient of the third due to first species are identical. This is illustrated in Figure 2.

Case 3: It is noticed that initially the third species is dominated by first up to time instant $t^* = 0.12$ and the second up to $t^* = 0.5$ after these dominate times we find reversal of the dominance. The birth rate of the second and the natural birth rate of both third species are almost equal. Further, it is evident that all the three species asymptotically converge to the equilibrium point. (Figure 3).

Case 4: In this case initially the first and second species dominates over the third species till the same time instant $t^* = 0.71$ after which the dominance is reversed. The initial values of the first, second and third species are in descending order. Further, the self inhibition coefficients of the first and third species are same as the natural birth rate of the third species. (Figure 4).

Case 5: The initial values of all the first, second and third species are identical. The natural birth rates of the first and second species are almost same. It is noticed that the third species is dominated by the second which itself dominated by the first as shown in Figure 5.

Case 6: Initially the first species dominates over the third species till the time instant $t^* = 0.43$ after which the dominance is reversed. The first species with low natural birth rate. Further, it is evident that the natural growth rate of the second species and the interaction coefficient of the first due to third species are identical.

Case 7: The second species has the least natural birth rate. The initial value of the second species is greater than other two. Further, it is evident that all the three species asymptotically converge to the equilibrium point. (Figure 7).

Case 8: In this case initially the second species dominates over the first till the time instant $t^* = 0.25$ after which the dominance is reversed. Further, we notice that all the three species are a weak competitor with no appreciable growth from some time instant. (Figure 8).

CONCLUSION

The present paper deals with an investigation on the stability of athree species syn eco-system with limited resources. In this paper we established all possible equilibrium states. It is conclude that, in all eight equilibrium states, only the normal steady state is stable. Further the global stability is established with the help of suitable Liapunov's function and the numerical solutions are computed using Runge-Kutta fourth order method.

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