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A Study On Multi Ecological System Consisting Of Three Species A Prey, Predator And Neutralism

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Abstract

Predation is the interaction where one species gets benefits at the expense of the other. In this interaction between two organisms one organism captures bio-mass from another. This is called as predator. In this interaction one organism eats away the another with its closeness of association. The lack of interaction between components of a mixed organism is known as neutralism. Despite coexisting, the species do not know one another, do not hurt one another, and do not benefit one another. In this paper, we study on multi ecological system consisting of three species (S_1, S_2, S_3) a prey, predator and neutralism with mortality rates. Here S_1 is the prey of two predators (S_2, S_3), the second and third species are neutrals. Three simultaneous non-linear first ordinary differential equations make up the mathematical model equations. There are recognized criteria for each of the eight equilibrium states' asymptotic stability. When all real roots are found to be negative and for complex roots having negative real part, the system become stable. Illustrations are provided for the perturbation trajectories over the equilibrium states. Additionally, the system's global stability is achieved by appropriately establishing Liapunov's function.

Keywords: Characteristic Equation, Competition, Equilibrium Point, Stable, Unstable.

1. Introduction:

Ecology is the study of how living things interact with their surroundings. Plants and animals are considered creatures, while the surroundings of animals are considered part of the environment. The study of living things (plants and animals) in connection to their environments and habits is known as ecology. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution- prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and synecology (the study of two or more communities). Synecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. This area of knowledge seeks to explain how many different kinds of plants and animals can live together in the same place for many generations. Animals and plants share the same habitat. Sometimes they can only share for so long before some locally go extinct, but there are other circumstances when many different kinds persist in a habitat indefinitely. As such, ecology may also be referred as the study of distribution and abundance of species under habitat availing the same resources. The Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on. Important studies in the field of theoretical ecology have been reviewed by Kot [4] and Gillman [3]. Numerous mathematicians and ecologists made contributions to the development of this field of study. The two primary sub-divisions of mathematical ecology, Autecology and Synecology, are separated into larger categories and are discussed in the works of Anna Sher [1], Arumugam [2], and Sharma [21]. Understanding the reciprocal interactions (positive or negative) between the interacting species is mostly dependent on mathematical modelling. Ma [6], Moghadas [7], Murray [8], and Sze-Bi Hsu [23] are among the writers who were first exposed to the broad ideas of modelling in biological science. The competitive ecosystem of two and three species with finite and infinite resources was investigated by Srinivas [22]. Narayan [9] went on to study prey-predator ecological models in which the predator had alternate food sources and the victim had partial shelter. Additionally, Kumar [5] investigated a few mathematical ecological commensalism models. Prasad [10–20], the current author, studied discrete and continuous models on syn-ecosystems with two, three, and four species.

2. Basic Equations of the Model:

The set of non-linear first order ordinary differential equations that follows, using the following symbols, provides the model equations for the rivalry between the three species.

Notation Adopted:

$N_i(t)$: $S_i's$ population strength at time t, i =1, 2 , 3.

t : Time period.

d_i : $S_i's$ natural death rate, i =1,2,3.

a_{ii} : $S_i's$ self- inhibition coefficients, i =1,2,3.

a_{12}, a_{21} : Interaction coefficients of S_1 due to S_2 and S_2 due to S_1.

a_{13}, a_{31} : Interaction coefficients of S_1 due to S_3 and S_3 due to S_1.

$e_i = \frac{d_i}{a_{ii}}$: Extinction coefficient of S_i, i =1,2,3 .

Additionally, it is expected that the model constants $d_1, d_2, d_3, a_{11}, a_{22}, a_{33}, e_1, e_2, e_3$, a_{12}, a_{21}, a_{13}, a_{31} are either zero or positive, and that the variables, N_1, N_2 & N_3, are either zero or positive.

Governing equations for the three species ecological system:

$$\frac{dN_1}{dt} = -(d_1N_1 + a_{11}N_1^2 + a_{12}N_1N_2 + a_{13}N_1N_3) \quad (1)$$

$$\frac{dN_2}{dt} = -d_2N_2 - a_{22}N_2^2 + a_{21}N_1N_2 \quad (2)$$

$$\frac{dN_3}{dt} = -d_3N_3 - a_{33}N_3^2 + a_{31}N_1N_3 \quad (3)$$

3. Critical Points

The eight critical points of the investigation are presented by

$$\frac{dN_i}{dt} = 0; i = 1,2,3 \quad (4)$$

Totally washed state:

$$E_1 : \overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = 0$$

Points in which only two are washed out while the other is not:

$$E_2 : \overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = -e_3$$

$$E_3 : \overline{N_1} = 0, \overline{N_2} = -e_2, \overline{N_3} = 0$$

$$E_4 : \overline{N_1} = -e_1, \overline{N_2} = 0, \overline{N_3} = 0$$

Points where only one is washed out and others are not.:

$E_5 : \overline{N_1} = 0, \overline{N_2} = -e_2, \overline{N_3} = -e_3$

$E_6 : \overline{N_1} = \mu_1, \overline{N_2} = 0, \overline{N_3} = \mu_2$; where $\mu_1 = \dfrac{d_3a_{13} - d_1a_{33}}{a_{13}a_{31} + a_{33}a_{11}}$, $\mu_2 = -\dfrac{a_1a_{31} + a_3a_{11}}{a_{13}a_{31} + a_{33}a_{11}}$

$E_7 : \overline{N_1} = \mu_3, \overline{N_2} = \mu_4, \overline{N_3} = 0$; where $\mu_3 = \dfrac{d_2a_{12} - d_1a_{22}}{a_{11}a_{22} + a_{21}a_{12}}$, $\mu_4 = -\dfrac{d_1a_{21} + d_2a_{11}}{a_{11}a_{22} + a_{21}a_{12}}$

The normal steady state:

$E_8 : \overline{N_1} = \mu_5, \overline{N_2} = \mu_6, \overline{N_3} = \mu_7$; where $\mu_5 = \dfrac{d_2a_{12}a_{33} + d_3a_{13}a_{22} - d_1a_{22}a_{33}}{a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}}$,

$\mu_6 = \dfrac{d_3a_{13}a_{21} - d_2a_{11}a_{33} - d_1a_{21}a_{33} - d_2a_{13}a_{31}}{a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}}$, $\mu_7 = \dfrac{d_2a_{12}a_{31} - d_3a_{11}a_{22} - a_{12}a_{21}d_3 - d_1a_{22}a_{31}}{a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} + a_{13}a_{31}a_{22}}$

4. Stability of Critical Points:

Let $N_i(t) = (N_1, N_2, N_3) = \overline{N_i} + U_i(t); i = 1, 2, 3$

Where $U_i(t)$ is a small change to $N = (\overline{N_1}, \overline{N_2}, \overline{N_3})$. Fundamental equations are quasi lineralized for perturbed state as

$$\frac{du_1}{dt} = -(d_1 + 2a_{11}\overline{N_1} + a_{12}\overline{N_2} + a_{13}\overline{N_3})u_1 - a_{12}\overline{N_1}u_2 - a_{13}\overline{N_1}u_3 \quad (5)$$

$$\frac{du_2}{dt} = a_{21}\overline{N_2}u_1 + (-d_2 - 2a_{22}\overline{N_2} + a_{21}\overline{N_1})u_2 \quad (6)$$

$$\frac{du_3}{dt} = a_{31}\overline{N_3}u_1 + (-d_3 - 2a_{33}\overline{N_3} + a_{31}\overline{N_1})u_3 \quad (7)$$

The characteristic equation is $|A - \lambda I| = 0$

Where

$$A = \begin{pmatrix} -d_1 - 2a_{11}\overline{N_1} - a_{12}\overline{N_2} - a_{13}\overline{N_3} & -a_{12}\overline{N_1} & -a_{13}\overline{N_1} \\ a_{21}\overline{N_2} & -d_2 - 2a_{22}\overline{N_2} + a_{21}\overline{N_1} & 0 \\ a_{31}\overline{N_3} & 0 & -d_3 - 2a_{33}\overline{N_3} + a_{31}\overline{N_1} \end{pmatrix} \quad (8)$$

The critical state will be stable when all roots of (8) are negative (for real roots) or have negative real part (for complex roots).

4.1 Stability for fully washed out state (E_1):

The basic equations are linearized and obtained as

$$\frac{du_1}{dt} = -d_1u_1; \ \frac{du_2}{dt} = -d_2u_2; \frac{du_3}{dt} = -d_3u_3 \quad (9)$$

Latent equation of (9) is $(\lambda + d_1)(\lambda + d_2)(\lambda + d_3) = 0$ (10)

Its roots are $-d_1, -d_2, -d_3$

Obviously, all roots are negative, hence E_1 stable.

The solutions to equation (9) are obtained as

$$u_1 = u_{10}e^{-d_1t}, u_2 = u_{20}e^{-d_2t}, u_3 = u_{30}e^{-d_3t} \quad (11)$$

Here the initial values of u_1, u_2, u_3 are respectively u_{10}, u_{20}, u_{30}.

Trajectories of perturbations:

$\left(\dfrac{u_1}{u_{10}}\right)^{d_2} = \left(\dfrac{u_2}{u_{20}}\right)^{d_1}; \left(\dfrac{u_1}{u_{10}}\right)^{d_3} = \left(\dfrac{u_3}{u_{30}}\right)^{d_1}; \left(\dfrac{u_2}{u_{20}}\right)^{d_3} = \left(\dfrac{u_3}{u_{30}}\right)^{d_2}$ will be the trajectories in $u_1 - u_2$ plane

4.2 Stability of $E_2 : \overline{N_1} = 0, \overline{N_2} = 0, \overline{N_3} = -e_3$:

The linearized equations of this state is

$$\frac{du_1}{dt} = \beta u_1; \frac{du_2}{dt} = -d_2 u_2; \frac{du_3}{dt} = d_3 u_3 + \alpha u_1 \quad (12)$$

where $\beta = e_3 a_{13} - d_1$ and $\alpha = -e_3 a_{31}$

The characteristic equation is $(\lambda + d_1)(\lambda + d_2)(\lambda - d_3) = 0$ (13)

$-d_1, -d_2, d_3$ are the characteristic roots of (13)

Since one root is positive and other roots are negative, E_2 is not perfectly stable.
The solutions of (12) are,

$$u_1 = u_{10} e^{\beta t} , \ u_2 = u_{20} e^{-d_2 t} , \ u_3 = u_{30} e^{d_3 t} + \gamma \left(e^{\beta t} - e^{d_3 t} \right) \quad (14)$$

where γ is constant

Trajectories of perturbations:

$$\left(\frac{u_1}{u_{10}}\right)^{-d_2} = \left(\frac{u_2}{u_{20}}\right)^{\beta}; \left(\frac{u_1}{u_{10}}\right)^{\beta} = \left(\frac{u_3}{u_{30}}\right)^{-d_1}; \left(\frac{u_2}{u_{20}}\right)^{\beta} = \left(\frac{u_3}{u_{30}}\right)^{-d_2}$$ is the trajectory

4.3 Stability of $E_3 : \overline{N_1} = 0, \overline{N_2} = -e_2, \overline{N_3} = 0$:

The Linearized equations are

$$\frac{du_1}{dt} = Au_1; \frac{du_2}{dt} = d_2 u_2 - a_{21} e_2 u_1; \frac{du_3}{dt} = -d_3 u_3 \quad (15)$$

where $A = a_{12} e_2 - d_1$

The latent equation is $(\lambda - A)(\lambda - d_2)(\lambda + d_3) = 0$ (16)

$A, d_2, -d_3$ are the characteristic roots of (16)

Since one root is positive, E_3 is not stable.
The equations (15) has the solution,

$$u_1 = u_{10} e^{At}; u_2 = u_{20} e^{d_2 t} + \delta \left(e^{At} - e^{d_2 t} \right); u_3 = u_{30} e^{-d_3 t} \quad (17)$$

where δ is constant.

Trajectories of perturbations:

$$\left(\frac{u_1}{u_{10}}\right)^{-d_3} = \left(\frac{u_3}{u_{30}}\right)^{A}; \left(\frac{u_1}{u_{10}}\right)^{d_2} = \left(\frac{u_2 - \eta u_1}{u_{20} - \tau}\right)^{A}; \left(\frac{u_3}{u_{30}}\right)^{d_2} = \left(\frac{u_2 - \eta u_1}{u_{20} - \tau}\right)^{-d_3}$$ is the trajectory

Where η, τ are constants

4.4 Stability of $E_4 : \overline{N_1} = -e_1, \overline{N_2} = 0, \overline{N_3} = 0$:

The Linearized equations are

$$\frac{du_1}{dt} = d_1 u_1 + a_{12} e_1 u_2 + a_{13} e_1 u_3; \frac{du_3}{dt} = Du_3; \frac{du_2}{dt} = -(d_2 + a_{21} e_1) u_2 \quad (18)$$

where $D = -d_3 - a_{31} e_1$ (19)

$d_1, -d_2 - a_{21} e_1, -d_3 - a_{31} e_1$ are the characteristic roots

Since one root is positive, hence E_4 is unstable.
The equations (18) obtained the solutions as

$$u_1 = \beta_1 e^{d_1 t} + \alpha_1 e^{-d_2 t} + \alpha_2 e^{Dt}; u_2 = u_{20} e^{-(d_2 + a_{21} e_1)t}; u_3 = u_{30} e^{Dt} \quad (20)$$

where α_1, α_2 is constants and $\beta_1 = u_{10} + \alpha_1 - \alpha_2$

Trajectories of perturbations:

$$\left(\frac{u_2}{u_{20}}\right)^D = \left(\frac{u_3}{u_{30}}\right)^{-d_2 - a_{21}e_1} ; \left(\frac{u_2}{u_{20}}\right)^{d_1} = \left(u_1 - \zeta_2 u_2 - \chi_3 u_3\right)^{-d_2 - a_{21}e_1} ; \left(\frac{u_3}{u_{30}}\right)^{d_1} = \left(u_1 - \zeta_2 u_2 - \chi_3 u_3\right)^D$$

is the trajectory. Where ζ_2, χ_3 are constants

4.5 Stability of $E_5 : \overline{N_1} = 0, \overline{N_2} = -e_2, \overline{N_3} = -e_3$:

The Linearized equations are

$$\frac{du_1}{dt} = \delta_1 u_1 \text{ ;where } \delta_1 = (a_{12}e_2 + a_{13}e_3) - d_1$$

$$\frac{du_2}{dt} = d_2 u_2 - a_{21}e_2 u_1 \text{ , } \frac{du_3}{dt} = d_3 u_3 - a_{31}e_3 u_1 \quad (21)$$

The characteristic equation is $(\lambda - \delta_1)(\lambda - d_2)(\lambda - d_3) = 0$ (22)

Since two roots are positive, E_5 is unstable.

The equations (21) has the following solution,

$$u_1 = u_{10} e^{\delta_1 t} ; u_2 = (u_{20} - \sigma_1) e^{d_2 t} + \sigma_1 e^{\delta_1 t} ; u_3 = (u_{30} - \sigma_2) e^{d_3 t} + \sigma_2 e^{\delta_1 t} \quad (23)$$

where σ_1, σ_2 are constants and $\delta_1 = a_{12}e_2 + a_{13}e_3 - d_1$

Trajectories of perturbations:

$$\left(\frac{u_1}{u_{10}}\right)^{d_2} = \left(\frac{u_2 - \eta_1}{u_{30} - \sigma_1}\right)^{\delta_1} ; \left(\frac{u_1}{u_{10}}\right)^{d_3} = \left(\frac{u_3 - \eta_2}{u_{30} - \sigma_2}\right)^{\delta_1} ; \left(\frac{u_2 - \eta_1}{u_{20} - \sigma_1}\right)^{d_3} = \left(\frac{u_3 - \eta_2}{u_{30} - \sigma_2}\right)^{d_2}$$

is the trajectory where η_1, η_2 are constants

5.6 Stability of $E_6 : \overline{N_1} = \mu_1, \overline{N_2} = 0, \overline{N_3} = \mu_2$:

The Linearized equations are

$$\frac{du_2}{dt} = \psi_1 u_2 \text{ ;where } \psi_1 = a_{21}\mu_1 - d_2$$

$$\frac{du_1}{dt} = E_1 u_1 - E_2 u_3 - E_3 e^{\psi_1 t} \text{ ;where } E_1 = -d_1 - 2a_{11}\mu_1 - a_{13}\mu_2 ; E_2 = a_{13}\mu_1 ; E_3 = a_{12}\mu_1 u_{20}$$

$$\frac{du_3}{dt} = E_4 u_3 + E_5 u_1 \text{ ;where } E_4 = -d_3 - 2a_{33}\mu_2 + a_{31}\mu_1 ; E_5 = a_{31}\mu_2 \quad (24)$$

The characteristic equation is cubic equation in which one root is $a_{21}\mu_1 - d_2$ and λ_1, λ_2 are the positive roots of the below equation

$$\lambda^2 - \lambda(E_1 + E_4) + (E_1 E_4 + E_2 E_5) = 0 \quad (25)$$

Hence, E_6 is unstable.

The equations (24) yield the solutions,

$$u_1 = \tau_4 e^{F_3 t} + \tau_5 e^{F_5 t} + \tau_6 e^{\psi_1 t} ; u_2 = u_{20} e^{\phi_1 t} ; u_3 = \tau_1 e^{F_1 t} + \tau_2 e^{F_2 t} - \tau_3 e^{\psi_1 t} \quad (26)$$

where $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, F_1, F_2, F_3, F_5$ are constants

Trajectories of perturbations:

$$u_3 + \tau_3\left(\frac{u_2}{u_{20}}\right) = \tau_1\left(\frac{u_2}{u_{20}}\right)^{\rho_1} + \tau_2\left(\frac{u_2}{u_{20}}\right)^{\rho_2} \text{ and } u_1 - \tau_6\left(\frac{u_2}{u_{20}}\right) = \tau_4\left(\frac{u_2}{u_{20}}\right)^{\rho_3} + \tau_5\left(\frac{u_2}{u_{20}}\right)^{\rho_4}$$

is the trajectory where $\rho_1, \rho_2, \rho_3, \rho_4$ are constants

4.7 Stability of $E_7 : \overline{N_1} = \mu_3, \overline{N_2} = \mu_4, \overline{N_3} = 0$:

The Linearized equations are

$$\frac{du_3}{dt} = \psi_2 u_3 \text{ ;where } \psi_2 = a_{31}\mu_3 - d_3$$

$$\frac{du_1}{dt} = E_6 u_1 - E_7 u_2 - E_8 u_3 \text{ ;where } E_6 = -(d_1 + 2a_{11}\mu_3 + a_{12}\mu_4); E_7 = a_{12}\mu_3; E_8 = a_{13}\mu_3$$

$$\frac{du_2}{dt} = E_9 u_2 + E_{10} u_1 \text{ ;where } E_9 = -(d_2 + 2a_{22}\mu_4) + a_{21}\mu_3; E_{10} = a_{21}\mu_4 \quad (27)$$

The characteristic equation is

$$(\lambda - \psi_2)\left[\lambda^2 + (E_6 + E_9)\lambda + (E_6 E_9 + a_{13}a_{21}\mu_3\mu_4)\right] = 0 \quad (28)$$

One root is ψ_2 and other roots noted to be negative.

Case(i): If ψ_2 is negative, E_7 is stable.

Case(ii): When ψ_2 is positive, E_7 is unstable.

Case(ii): If ψ_2 is zero, E_7 is neutrally stable.

The equations (24) yield the solutions

$$u_1 = \omega_4 e^{F_7 t} + \omega_5 e^{F_8 t} + \omega_6 e^{\psi_2 t}; u_2 = \omega_1 e^{F_5 t} + \omega_2 e^{F_6 t} - \omega_3 e^{\psi_2 t}; u_3 = u_{30} e^{-e_2 t} \quad (29)$$

where $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, F_5, F_6, F_7, F_8$ are constants

Trajectories of perturbations:

$u_2 + \omega_3\left(\frac{u_3}{u_{30}}\right) = \omega_1\left(\frac{u_3}{u_{30}}\right)^{\theta_1} + \omega_2\left(\frac{u_3}{u_{30}}\right)^{\theta_2}$ and $u_1 - \omega_6\left(\frac{u_3}{u_{30}}\right) = \omega_4\left(\frac{u_3}{u_{30}}\right)^{\theta_3} + \omega_5\left(\frac{u_3}{u_{30}}\right)^{\theta_4}$ is the trajectory where $\theta_1, \theta_2, \theta_3, \theta_4$ are constants

4.8 Stability to co-existent state (E_8) :

The Linearized equations to this state are

$$\frac{du_1}{dt} = -Au_1 - Bu_2 - Cu_3 \text{ ;where } A = a_{11}\mu_5; B = a_{12}\mu_5; C = a_{13}\mu_5$$

$$\frac{du_2}{dt} = Eu_1 - Fu_2 \text{ ;where } E = a_{21}\mu_6; F = a_{22}\mu_6$$

$$\frac{du_3}{dt} = Gu_1 - Fu_3 \text{ ;where } G = a_{31}\mu_7; H = a_{33}\mu_7 \quad (30)$$

The characteristic equation is $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$

$b_1 = A + F + H; b_2 = F + AH + BE + CG; b_3 = AFH + BEH + CFG$

According to Routh-Hurwitz's criteria, $b_1 > 0$, $b_3 > 0$ and $b_3(b_1 b_2 - b_3) > 0$.
Hence the co-existent state is **locally asymptotically stable**.
The equations yield the solutions,

$$u_1 = l_4 e^{-At} - \left[l_1 e^{-\lambda_1 t} + l_2 e^{-\lambda_2 t} + l_3 e^{-\lambda_3 t}\right]$$

$$u_2 = w_1 e^{-\lambda_1 t} + w_2 e^{-\lambda_2 t} + w_3 e^{-\lambda_3 t} \text{ and } u_3 = z_1 e^{-\lambda_1 t} + z_2 e^{-\lambda_2 t} + z_3 e^{-\lambda_3 t} \quad (31)$$

Where $A, l_1, l_2, l_3, w_1, w_2, w_3, z_1, z_2, z_3$ are constants.

Trajectories of perturbations:

In $u_1 - u_2 - u_3$ plane the trajectory is given by

$$\left[\frac{u_1\rho_7 + l_2\rho_4 - l_3\rho_5}{-l_1\rho_1 + l_2\rho_2 - l_3\rho_3}\right]^{\lambda_2} = \left[\frac{-l_1\rho_4 - \rho_2\rho_7 - l_3\rho_6}{-l_1\rho_1 + l_2\rho_2 - l_3\rho_3}\right]^{\lambda_1},$$

$$\left[\frac{u_1\rho_7+l_2\rho_4-l_3\rho_5}{-l_1\rho_1+l_2\rho_2-l_3\rho_3}\right]^{\lambda_3}=\left[\frac{-l_1\rho_8+l_2\rho_6+\rho_7\rho_3}{-l_1\rho_1+l_2\rho_2-l_3\rho_3}\right]^{\lambda_1} \text{ and}$$

$$\left[\frac{-l_1\rho_4-\rho_2\rho_7-l_3\rho_6}{-l_1\rho_1+l_2\rho_2-l_3\rho_3}\right]^{\lambda_3}=\left[\frac{-l_1\rho_8+l_2\rho_6+\rho_7\rho_3}{-l_1\rho_1+l_2\rho_2-l_3\rho_3}\right]^{\lambda_1} \text{ where } \rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8$$

are constants and $\rho_7=u_1-l_4e^{-At}$

5. Liaupnov's Function for Global Stability

We have covered the stability at local stage, each of the eight equilibrium states in section 4. whereas the other states are unstable and the E_1, E_7 and E_8 are stable. Using the appropriate Lyapunov's functions, now we investigate the global stability of dynamical systems (1), (2), and (3) at these stages.

Theorem 1: Asymptotically, the equilibrium state E_7 is universally stable.
Proof : Let us consider the following Liaupunov's function for three species

$$L(N_1,N_2)=\left(N_1-\overline{N_1}-\overline{N_1}\ln\left(\frac{N_1}{\overline{N_1}}\right)\right)+l_1\left(N_2-\overline{N_2}-\overline{N_1}\ln\left(\frac{N_2}{\overline{N_2}}\right)\right) \quad (32)$$

where l_1 are constants to be chosen. By the time derivative, we get

$$\frac{dL}{dt}=\left(\frac{N_1-\overline{N_1}}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\overline{N_2}}{N_2}\right)\frac{dN_2}{dt}$$

$$=\left(N_1-\overline{N_1}\right)\left(-d_1-a_{11}N_1-a_{12}N_2\right)+l_1\left(N_2-\overline{N}_2\right)\left(-d_2-a_{22}N_2+a_{21}N_1\right)$$

$$\frac{dL}{dt}=-a_{11}\left(N_1-\overline{N_1}\right)^2-l_1a_{22}\left(N_2-\overline{N_2}\right)^2+\left(-a_{12}+l_1a_{21}\right)\left(N_1-\overline{N_1}\right)\left(N_2-\overline{N_2}\right) \quad (33)$$

The constant l_1 which are positive are to be chosen such that coefficient of $\left(N_1-\overline{N_1}\right)\left(N_2-\overline{N_2}\right)$ in (33) to vanish.

With this choice of constant l_1, we get $l_1=\frac{a_{12}}{a_{21}}$

Thus, we get $\frac{dL}{dt}$ is negative definite.

Therefore, equilibrium state is "stable"globally asymptotically.
Theorem 2: The normal steady state is balanced universally.
Proof : Revolve the mapping,

$$L(N_1,N_2,N_3)=\left(N_1-\overline{N_1}-\overline{N_1}\ln\left(\frac{N_1}{\overline{N_1}}\right)\right)+l_1\left(N_2-\overline{N_2}-\overline{N_1}\ln\left(\frac{N_2}{\overline{N_2}}\right)\right)+l_2\left(N_3-\overline{N_3}-\overline{N_3}\ln\left(\frac{N_3}{\overline{N_3}}\right)\right) \text{ where}$$

l_1,l_2 are constants to be chosen.

By the time derivative, we get

$$\frac{dL}{dt}=\left(\frac{N_1-\overline{N_1}}{N_1}\right)\frac{dN_1}{dt}+l_1\left(\frac{N_2-\overline{N_2}}{N_2}\right)\frac{dN_2}{dt}+l_2\left(\frac{N_3-\overline{N_3}}{N_3}\right)\frac{dN_3}{dt}$$

$$=\left(N_1-\overline{N_1}\right)\left(a_1-a_{11}N_1-a_{12}N_2-a_{13}N_3\right)+l_1\left(N_2-\overline{N}_2\right)\left(a_2-a_{22}N_2+a_{21}N_1\right)+$$

$$l_2\left(N_3-\overline{N}_3\right)\left(a_3-a_{33}N_3+a_{31}N_1\right)$$

$$\frac{dL}{dt}=-a_{11}\left(N_1-\overline{N_1}\right)^2-l_1a_{22}\left(N_2-\overline{N_2}\right)^2-l_2a_{33}\left(N_3-\overline{N_3}\right)^2+\left(N_1-\overline{N_1}\right)\left(N_2-\overline{N_2}\right)\left(-a_{12}+l_1a_{21}\right)+$$

$$\left(N_1-\overline{N_1}\right)\left(N_2-\overline{N_2}\right)\left(-a_{13}+l_2a_{31}\right) \quad (34)$$

The constant l_1, l_2 which are positive are to be chosen such that coefficient of $\left(N_1 - \overline{N_1}\right)\left(N_2 - \overline{N_2}\right)$ and $\left(N_1 - \overline{N_1}\right)\left(N_3 - \overline{N_3}\right)$ in (34) to vanish.

With this choice of constant l_1, l_2 from (34) we get $l_1 = \frac{a_{12}}{a_{21}}, l_2 = \frac{a_{13}}{a_{31}}$ and $l_1 a_{22} > 0, l_2 a_{33} > 0$.

Thus, we get $\frac{dL}{dt} = -\left[a_{11}\left(N_1 - \overline{N_1}\right)^2 + l_1 a_{22}\left(N_2 - \overline{N_2}\right)^2 + l_2 a_{33}\left(N_3 - \overline{N_3}\right)^2 \right]$ and is negative definite.

Therefore, equilibrium state is “stable”globally asymptotically.

6. Conclusion

In this present paper, we study on multi ecological system consisting of three species a prey, predator and neutralism with mortality rates. All possible equilibrium states for the model are identified and stability for criteria is discussed. It has been observed in all equilibrium states that, only fully washed out state, normal steady state and the equilibrium E_7 are stable. Additionally, using a properly constructed Liapunov's function, there has been established a global stability of the system, and the trajectories of the perturbations across the equilibrium points are depicted.

References

1. A. Anna Sher and Jr. Molles, Manuel C , *Ecology: Concepts & Applications*, McGraw-Hill, New York, 2018.
2. N. Arumugam, *Concepts of Ecology*, Saras Publication, India, 2006.
3. M. Gillman, *An Introduction to Mathematical Models in Ecology and Evolution*, Wiley-Blackwell, U.K., 2009.
4. M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, U.K, 2001.
5. N.P. Kumar, *Some Mathematical Models of Ecological Commensalism*, Acharya Nagarjuna University, Ph.D. Thesis. 2010.
6. Z. Ma, *The Study of Biology Model*, Anhui Education Press, China, 1996.
7. M. Moghadas Seyed and Majid Jaberi-Douraki, *Mathematical Modelling*, John Wiley & Sons, New Jersey, United States, 2018.
8. J.D. Murray, *Mathematical Biology*, Springer, New York, 1993.
9. K.L. Narayan, *A Mathematical Study of Prey-Predator Ecological Models with a Partial Cover for the Prey and Alternative Food for the Predator*, J.N.T.University, Ph.D Thesis, 2004.
10. B.H. Prasad, Stability Analysis of a Three Species Non-Linear Eco-System with Restricted Resources, *The Journal of Indian Mathematical Society*, 2019, **86**: 126-136.
11. B.H. Prasad, Mathematical Modeling on Ecosystem Consisting of Two Hosts and One Commensal with Mortality Rates for the First and Third Species, *World Journal of Modelling and Simulation*, 2017, **13(3)**: 163-172.
12. B.H. Prasad, A Study on Discrete Model of a Prey-Predator Eco-System with Limited and Unlimited Resources, *The Journal of Indian Mathematical Society*, 2015, **82(3-4)** : 169-179.
13. B.H. Prasad, A Study on the Discrete Modeling of Three Species Syn-Eco-System with Unlimited Resources, *Journal of Applied Mathematics and Computational Mechanics*, 2015, **14** : 85-93.
14. B.H. Prasad, A Discrete Model of a Typical Three Species Syn- Eco – System with Unlimited Resources for the First and Third Species, *Asian Academic Research Journal of Multidisciplinary*, 2014, **1(17)** : 36-46.
15. B.H. Prasad, A Study on Discrete Model of Three Species Syn-Eco-System with Limited Resources, *Int. Journal of Modern Education and Computer Science*, 2014, **11** : 38-44.
16. B.H. Prasad, The Stability Analysis of a Three Species Syn-Eco-System with Mortality Rates, *Contemporary Mathematics and Statistics*, 2014, **2(2)**: 76-89.
17. B.H. Prasad and N. Ch. P. Ramacharyulu, Global Stability of Four Species Syn Eco-System: A Prey-Predator-Host-Commensal- A Numerical Approach, *Int. J. of Mathematics and Computer Applications Research*, 2013, **3**: 275-290.
18. B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a typical three species syn eco-system, *Int. Journal of Mathematical Archive*, 2012, **3(10)** : 3583-3601.

19. B.H. Prasad and N. Ch. P. Ramacharyulu, Discrete Model of Commensalism Between Two Species, *International Journal of Modern Education and Computer Science*, 2012, **8** : 40-46.
20. B.H. Prasad and N. Ch. P. Ramacharyulu, On the Stability of a Four Species Syn Eco-System with Commensal Prey Predator Pair with Prey Predator Pair of Hosts – VII (Host of S2 washed out states), *Journal of Communication and Computer*, 2011, **8(6)** : 415-421.
21. P.D. Sharma, *Ecology and Environment*, Rastogi Publications, India, 2009.
22. N.C. Srinivas, *Some Mathematical Aspects of Modeling in Bio-medical Sciences*, Kakatiya University, Ph.D Thesis, 1991.
23. [Sze-Bi Hsu, *Mathematical Modeling in Biological Science*, Tsing-Hua University, Taiwan, 2004.