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An Analysis Of Stability Behaviour For Two Preys And Two Predators Ecological Model

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Abstract

This study investigates the dynamic interactions among four species within an ecological framework. The population model describes the cooperative dynamics between two preys and two predators as these species cooperate to fulfill various needs, including food requirements and shelter. Simultaneously, interspecific predation may emerge as distinct species compete for survival within the same habitat. The mathematical model comprises a set of nonlinear differential equations for each prey and predator. A comprehensive exploration of feasible equilibrium positions is undertaken to assess species stability at various stages, considering factors such as positivity and boundedness. Furthermore, the model employs the Routh-Hurwitz Criterion and Lyapunov function to investigate both local and global coexistence states of the species.

Keywords: Prey, Predator, Equilibrium Positions, Positivity, Boundedness, Routh-Hurwitz Criterion, Lyapunov Function.

1. Introduction

In the field of ecology, studying changing populations of species is crucial for understanding the intricate dynamics of life within ecosystems. In this context, species, serving as the foundational elements of ecological systems, exhibit unique characteristics while carving out specialized niches within their environments. The dynamic nature of species populations encompasses the patterns and processes governing their growth, distribution, interactions, and responses to environmental changes. The famous ecologist study of evolving species populations is central to ecology, providing insights into the complex interactions that drive ecosystems [2,10-12]. Ecological models serve as essential tools for scientists and researchers to understand and predict the dynamics of ecosystems, species interactions, and the effects of various factors on the natural world [4,5,7].

Ashok Mondal et.al., investigated the study about fear effect and harvesting effort between the prey predator species [6].M.Gunasekaran, et.al., construct and analyzed the optimal harvesting dynamic model, holling functional response with host ecosystem and eco-epidemiological prey predator model [15,17,18]. Ecologists analyze the complex dynamics of population fluctuations in various species over time, influenced by factors such as foraging, refuge, square root function, resource availability, fear effect, functional response and environmental conditions [3, 8, 9, 16, 20]. The complex web of interactions within ecological systems has long fascinated researchers, leading to the development of various models and frameworks aimed at understanding the dynamics of species, coexistence patterns, ecological consequences of interactions and adaptive protection mutualism [24, 25, 28, 31].

One such model that has garnered considerable attention within the field of ecology is the four species model. This model represents a structured and detailed approach to investigating the dynamics of ecosystems by focusing on the interactions and relationships among four individual or group of the same species. The four species ecological model is built on the principles of population dynamics, species contacts, fractional population, adaptive protection and resource competition [1,13,29]. It involves the study of dissimilar species interact with each other within an ecosystem, considering factors such as cooperation, predation, competition for resources, host-commensal and mutualistic relationships [21, 22, 26, 30].Researches investigate and analysis the species communication in ratio-dependent, bifurcation analysis, chaos control, fractional population competition and ecological consequences of these three and four species model [23, 27, 29].

Jinxing Zhao and Yuanfu Shao, examine the stochastic analysis between the prey predator species with distributed delay [14]. Many ecological researchers have conducted field studies to validate four species ecological models and to understand how they apply to real-world ecosystems. These studies often focus on specific ecosystems, such as coral reefs, forests, or aquatic systems. With the growing concern about climate change and its impact on ecosystems, researchers have used four species ecological models to assess how shifts in temperature, habitat loss, and other environmental changes affect the dynamics of species interactions. Furthermore, this investigation will assess the model's applications in understanding and predicting the responses of ecological communities to environmental perturbations, and offer practical implications for ecosystem management, conservation, and preservation [17-19]. It can help inform strategies for preserving biodiversity, managing invasive species, and restoring damaged ecosystems. As ecosystems around the world face unprecedented challenges, from climate change to habitat loss and invasive species, it becomes increasingly critical to delve into the complexities of species interactions.

2. Mathematical Model

The proposed ecological model describes four species, namely first prey (N_1), second prey (N_2), primary predator (N_3) and secondary predator (N_4). The prey species N_1 and N_2as well as the predator species N_3 and $N_4$are mutually assist each other within their respective groups. At the same time, natural interaction occurs between predator and prey species(N_3, N_4andN_1, N_2) for their food, shelter and other resources in the environment. The system examines the continued existence of prey while battling with other two species, and also the stable coexistence interaction between predator and Prey in their habitat. Sixteen equilibrium points are acknowledged and derived. Finally, the local and global stability of all the four species in existence state is depicted.

This model can be used to assess the stability of ecosystems, predict species abundances, and understand the consequences of environmental changes of these four species. This field of study goes beyond mere population size and explores the dynamics of species interactions, including cooperation, competition, predation, and mutualism. In essence, the exploration of species in the context of dynamic populations in ecology unveils the fascinating web of life's interconnections, shedding light on the ever-changing tapestry of ecosystems and the intricate processes that sustain them.

The model characterized by nonlinear differential equation for the given system as follows

$$\frac{dN_1}{dt} = a_1N_1 - b_{11}N_1^2 + b_{12}N_1N_2 - b_{13}N_1N_3 - b_{14}N_1N_4 \qquad \ldots(1)$$

$$\frac{dN_2}{dt} = a_2N_2 - b_{22}N_2^2 + b_{21}N_1N_2 - b_{23}N_2N_3 - b_{24}N_2N_4 \qquad \ldots(2)$$

$$\frac{dN_3}{dt} = a_3N_3 - b_{33}N_3^2 + b_{31}N_1N_3 + b_{32}N_2N_3 + b_{34}N_3N_4 \qquad \ldots(3)$$

$$\frac{dN_4}{dt} = a_4N_4 - b_{44}N_4^2 + b_{41}N_1N_4 + b_{42}N_2N_4 + b_{43}N_3N_4 \qquad \ldots(4)$$

The initial conditions for the populations are given as$N_1(0) > 0, N_2(0) > 0, N_3(0) > 0, N_4(0) > 0$, indicating that the initial populations of all four species are greater than zero. These equations characterize the changes in population sizes over time, incorporating factors such as growth rates and both intraspecific competition and interspecific interactions among the species.

3. The Symbolic Representation of the System of Equations

$N_i{}'s$–Population density of i^{th} species. (where i = 1,2,3,4).
$a_i{}'s$ – Natural growth rates of N_i's , (where i = 1,2,3,4).
$b_{ii}{}'s$ – Decrease rate of the species N_i's due to intraspecific competition. (where i = 1,2,3,4).
b_{12} &b_{21} –Increase rate of N_1&N_2 due to mutualism with each other.
b_{13} &b_{23} –Decrease rate of N_1&N_2 due to the inhibitory effect of N_3.
b_{14} &b_{24} –Decrease rate of N_1&N_2 due to inhibitory effect ofN_4.
b_{31} &b_{41} –Increase rate of N_3&N_4 due to predation with N_1.
b_{32} &b_{42} –Increase rate of N_3&N_4 due to predation with N_2.
b_{34} &b_{43} –Increase rate of N_3&N_4 due to mutualism with each other.

4. Equilibrium State

Equilibria and stability of model

In this section, we discuss the equilibrium points of the system with their existence conditions and analyze the stability of system near the equilibrium points. Now the sixteen equilibrium points of proposed system are as follows.

I. E_T : Trivial Equilibrium Point

$$E_T(0,0,0,0): \quad \overline{N}_1 = 0, \overline{N}_2 = 0, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = 0$$

II. E_A : The position in which one of the species exists and the other three are extinct.

$$E_{A1}(\overline{N}_1,0,0,0): \quad \overline{N}_1 = \frac{a_1}{b_{11}}, \quad \overline{N}_2 = 0, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = 0$$

$$E_{A2}(0,\overline{N}_2,0,0): \quad \overline{N}_1 = 0, \quad \overline{N}_2 = \frac{a_2}{b_{22}}, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = 0$$

$$E_{A3}(0,0,\overline{N}_3,0): \quad \overline{N}_1 = 0, \overline{N}_2 = 0, \quad \overline{N}_3 = \frac{a_3}{b_{33}}, \quad \overline{N}_4 = 0$$

$$E_{A4}(0,0,0,\overline{N}_4): \quad \overline{N}_1 = 0, \quad \overline{N}_2 = 0, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = \frac{a_4}{b_{44}}$$

III. E_S:The position in which the two species are surviving and the remaining two are extinct.

$$E_{S1}(\overline{N}_1,\overline{N}_2,0,0): \quad \overline{N}_1 = \frac{a_1b_{22} + a_2b_{12}}{b_{11}b_{22} - b_{12}b_{21}}, \quad \overline{N}_2 = \frac{a_2b_{11} + a_1b_{21}}{b_{11}b_{22} - b_{12}b_{21}}, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = 0$$

This state is exist only when $b_{11}b_{22} - b_{12}b_{21} > 0$

$$E_{S2}(0,\overline{N}_2,\overline{N}_3,0): \quad \overline{N}_1 = 0, \quad \overline{N}_2 = \frac{a_2b_{33} - a_3b_{23}}{b_{22}b_{33} + b_{23}b_{32}}, \quad \overline{N}_3 = \frac{a_3b_{22} + a_2b_{32}}{b_{22}b_{33} + b_{23}b_{32}}, \quad \overline{N}_4 = 0$$

$$E_{S3}(0,0,\overline{N}_3,\overline{N}_4): \quad \overline{N}_1 = 0, \quad \overline{N}_2 = 0, \quad \overline{N}_3 = \frac{a_3b_{44} + a_4b_{34}}{b_{33}b_{44} - b_{34}b_{43}}, \quad \overline{N}_4 = \frac{a_4b_{33} + a_3b_{43}}{b_{33}b_{44} - b_{34}b_{43}}$$

This state is exist only when $b_{33}b_{44} - b_{34}b_{43} > 0$

$$E_{S4}(\overline{N}_1,0,0,\overline{N}_4): \quad \overline{N}_1 = \frac{a_1b_{44} - a_4b_{14}}{b_{11}b_{44} + b_{14}b_{41}}, \overline{N}_2 = 0, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = \frac{a_4b_{11} + a_1b_{41}}{b_{11}b_{44} + b_{14}b_{41}}$$

$$E_{S5}(\overline{N}_1,0,\overline{N}_3,0): \quad \overline{N}_1 = \frac{a_1b_{33} - a_3b_{13}}{b_{11}b_{33} + b_{13}b_{31}}, \quad \overline{N}_2 = 0, \quad \overline{N}_3 = \frac{a_3b_{11} + a_1b_{31}}{b_{11}b_{33} + b_{13}b_{31}}, \quad \overline{N}_4 = 0$$

$$E_{S6}(0,\overline{N}_2,0,\overline{N}_4): \quad \overline{N}_1 = 0, \quad \overline{N}_2 = \frac{a_2b_{44} - a_4b_{24}}{b_{22}b_{44} + b_{24}b_{42}}, \quad \overline{N}_3 = 0, \quad \overline{N}_4 = \frac{a_4b_{22} + a_2b_{42}}{b_{22}b_{44} + b_{24}b_{42}}$$

IV. $E_{S7}(\overline{N}_1,\overline{N}_2,\overline{N}_3,0)$: The secondary predator free equilibrium point

$$\overline{N}_1 = \frac{a_1(b_{22}b_{33} + b_{23}b_{32}) + a_2(b_{12}b_{33} - b_{13}b_{32}) - a_3(b_{12}b_{23} + b_{22}b_{13})}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}(b_{23}b_{31} - b_{21}b_{33}) + b_{13}(b_{21}b_{32} + b_{31}b_{22})},$$

$$\overline{N}_2 = \frac{a_1(b_{21}b_{33} - b_{31}b_{23}) + a_2(b_{11}b_{33} + b_{13}b_{31}) - a_3(b_{11}b_{23} + b_{13}b_{21})}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}(b_{23}b_{31} - b_{21}b_{33}) + b_{13}(b_{21}b_{32} + b_{31}b_{22})},$$

$$\overline{N}_3 = \frac{a_1(b_{21}b_{32} + b_{22}b_{31}) + a_2(b_{11}b_{32} + b_{12}b_{31}) + a_3(b_{11}b_{22} + b_{12}b_{21})}{b_{11}(b_{22}b_{33} + b_{23}b_{32}) + b_{12}(b_{23}b_{31} - b_{21}b_{33}) + b_{13}(b_{21}b_{32} + b_{31}b_{22})}, \overline{N}_4 = 0$$

V. $E_{S8}(0,\overline{N}_2,\overline{N}_3,\overline{N}_4)$: The first prey free equilibrium point

$$\overline{N}_1 = 0, \quad \overline{N}_2 = \frac{a_2(b_{33}b_{44} - b_{34}b_{43}) - a_3(b_{23}b_{44} + b_{24}b_{43}) - a_4(b_{23}b_{34} + b_{24}b_{33})}{b_{22}(b_{33}b_{44} - b_{34}b_{43}) + b_{23}\left((b_{32}b_{44} + b_{34}b_{42})\right) + b_{24}(b_{32}b_{43} + b_{33}b_{42})},$$

$$\overline{N}_3=\frac{a_2(b_{32}b_{44}-b_{34}b_{42})+a_3(b_{22}b_{44}+b_{24}b_{42})-a_4(b_{24}b_{32}-b_{22}b_{34})}{b_{22}(b_{33}b_{44}-b_{34}b_{43})+b_{23}((b_{32}b_{44}+b_{34}b_{42}))+\ b_{24}(b_{32}b_{43}+b_{33}b_{42})},$$

$$\overline{N}_4=\frac{a_2(b_{32}b_{43}+b_{33}b_{42})-a_3(b_{23}b_{42}-b_{22}b_{43})+a_4(b_{22}b_{33}+b_{23}b_{32})}{b_{22}(b_{33}b_{44}-b_{34}b_{43})+b_{23}((b_{32}b_{44}+b_{34}b_{42}))+\ b_{24}(b_{32}b_{43}+b_{33}b_{42})}$$

VI. $E_{S9}(\overline{N}_1,0,\overline{N}_3,\overline{N}_4)$: The second prey free equilibrium point

$$\overline{N}_1=\frac{a_1(b_{33}b_{44}-b_{34}b_{43})-a_3(b_{13}b_{44}+b_{14}b_{43})-a_4(b_{13}b_{34}+b_{14}b_{33})}{b_{11}(b_{33}b_{44}-b_{34}b_{43})+b_{13}((b_{31}b_{44}+b_{34}b_{41}))+\ b_{14}(b_{31}b_{43}+b_{33}b_{41})},\overline{N}_2=0,$$

$$\overline{N}_3=\frac{a_1(b_{31}b_{44}-b_{34}b_{41})+a_3(b_{11}b_{44}+b_{14}b_{41})-a_4(b_{31}b_{14}-b_{11}b_{34})}{b_{11}(b_{33}b_{44}-b_{34}b_{43})+b_{13}((b_{31}b_{44}+b_{34}b_{41}))+\ b_{14}(b_{31}b_{43}+b_{33}b_{41})},$$

$$\overline{N}_4=\frac{a_1(b_{31}b_{43}+b_{33}b_{41})-a_3(b_{13}b_{41}-b_{11}b_{43})+a_4(b_{11}b_{33}+b_{13}b_{31})}{b_{11}(b_{33}b_{44}-b_{34}b_{43})+b_{13}((b_{31}b_{44}+b_{34}b_{41}))+\ b_{14}(b_{31}b_{43}+b_{33}b_{41})}$$

VII. $E_{S10}(\overline{N}_1,\overline{N}_2,0,\overline{N}_4)$: The primary predator free equilibrium point

$$\overline{N}_1=\frac{a_1(b_{22}b_{44}+b_{24}b_{42})+a_2(b_{12}b_{44}-b_{14}b_{42})-a_4(b_{12}b_{24}+b_{14}b_{22})}{b_{11}(b_{22}b_{44}+b_{24}b_{42})+b_{12}((b_{24}b_{41}-b_{21}b_{44}))+\ b_{14}(b_{21}b_{42}+b_{22}b_{41})}$$

$$\overline{N}_2=\frac{a_1(b_{21}b_{44}-b_{24}b_{41})+a_2(b_{11}b_{44}+b_{14}b_{41})-a_4(b_{11}b_{24}+b_{14}b_{21})}{b_{11}(b_{22}b_{44}+b_{24}b_{42})+b_{12}((b_{24}b_{41}-b_{21}b_{44}))+\ b_{14}(b_{21}b_{42}+b_{22}b_{41})},\overline{N}_3=0$$

$$\overline{N}_4=\frac{a_1(b_{21}b_{42}+b_{22}b_{41})+a_2(b_{11}b_{42}+b_{12}b_{41})+a_4(b_{11}b_{22}-b_{12}b_{21})}{b_{11}(b_{22}b_{44}+b_{24}b_{42})+b_{12}((b_{24}b_{41}-b_{21}b_{44}))+\ b_{14}(b_{21}b_{42}+b_{22}b_{41})}$$

VIII. $E_{P1}(\overline{N}_1,\overline{N}_2,\overline{N}_3,\overline{N}_4)$: Coexistence state for all the four species

$$\overline{N}_1=\frac{\begin{bmatrix} a_1[(b_{22}b_{33}b_{44}-b_{22}b_{34}b_{43})+(b_{23}b_{32}b_{44}+b_{23}b_{34}b_{42})+(b_{24}b_{32}b_{43}+b_{24}b_{33}b_{42})] \\ +a_2[(b_{12}b_{33}b_{44}-b_{12}b_{34}b_{43})-(b_{13}b_{32}b_{44}+b_{34}b_{42}b_{13})+(b_{14}b_{33}b_{42}-b_{14}b_{32}b_{43})] \\ +a_3[-(b_{12}b_{23}b_{44}+b_{12}b_{24}b_{43})-(b_{13}b_{22}b_{44}+b_{13}b_{24}b_{42})+(b_{14}b_{23}b_{42}-b_{14}b_{22}b_{43})] \\ +a_4[-(b_{12}b_{23}b_{34}+b_{12}b_{24}b_{33})+(b_{13}b_{24}b_{32}-b_{13}b_{22}b_{33})-(b_{14}b_{22}b_{33}+b_{14}b_{23}b_{32})] \end{bmatrix}}{\begin{bmatrix} b_{11}b_{22}b_{33}b_{44}-b_{11}b_{22}b_{34}b_{43}+\ b_{11}b_{23}b_{32}b_{44}+\ b_{11}b_{23}b_{34}b_{42}+b_{11}b_{24}b_{32}b_{43} \\ +b_{11}b_{24}b_{42}b_{33}-b_{12}b_{21}b_{33}b_{44}+\ b_{12}b_{21}b_{34}b_{43}+\ b_{12}b_{23}b_{31}b_{44}+b_{12}b_{23}b_{34}b_{41} \\ +b_{12}b_{24}b_{31}b_{43}+b_{12}b_{24}b_{33}b_{41}+b_{13}b_{21}b_{32}b_{44}+b_{13}b_{34}b_{42}b_{21}+b_{13}b_{22}b_{31}b_{44} \\ +b_{13}b_{22}b_{34}b_{41}+b_{13}b_{24}b_{31}b_{42}-b_{13}b_{24}b_{41}b_{32}+b_{14}b_{21}b_{32}b_{43}+b_{14}b_{21}b_{33}b_{42} \\ +\ b_{14}b_{22}b_{31}b_{43}+\ b_{14}b_{22}b_{33}b_{41}-b_{14}b_{23}b_{31}b_{42}+b_{14}b_{23}b_{32}b_{41} \end{bmatrix}}$$

$$\overline{N}_2=\frac{\begin{bmatrix} a_1[(b_{21}b_{33}b_{44}-b_{21}b_{34}b_{43})-(b_{23}b_{31}b_{44}+b_{23}b_{34}b_{41})-(b_{24}b_{31}b_{43}+b_{24}b_{33}b_{41})] \\ +a_2[(b_{11}b_{33}b_{44}-b_{11}b_{34}b_{43})+(b_{13}b_{31}b_{44}+b_{13}b_{34}b_{41})+(b_{14}b_{31}b_{43}+b_{14}b_{33}b_{41})] \\ +a_3[-(b_{11}b_{23}b_{44}+b_{11}b_{24}b_{43})+(b_{13}b_{24}b_{41}-b_{13}b_{21}b_{44})+(b_{14}b_{21}b_{43}+b_{14}b_{23}b_{41})] \\ +a_4[-(b_{11}b_{23}b_{34}+b_{11}b_{24}b_{33})-(b_{13}b_{21}b_{34}+b_{13}b_{24}b_{31})+(b_{14}b_{31}b_{23}-b_{14}b_{21}b_{33})] \end{bmatrix}}{\begin{bmatrix} b_{11}b_{22}b_{33}b_{44}-b_{11}b_{22}b_{34}b_{43}+\ b_{11}b_{23}b_{32}b_{44}+\ b_{11}b_{23}b_{34}b_{42}+b_{11}b_{24}b_{32}b_{43} \\ +b_{11}b_{24}b_{42}b_{33}-b_{12}b_{21}b_{33}b_{44}+\ b_{12}b_{21}b_{34}b_{43}+\ b_{12}b_{23}b_{31}b_{44}+b_{12}b_{23}b_{34}b_{41} \\ +b_{12}b_{24}b_{31}b_{43}+b_{12}b_{24}b_{33}b_{41}+b_{13}b_{21}b_{32}b_{44}+b_{13}b_{34}b_{42}b_{21}+b_{13}b_{22}b_{31}b_{44} \\ +b_{13}b_{22}b_{34}b_{41}+b_{13}b_{24}b_{31}b_{42}-b_{13}b_{24}b_{41}b_{32}+b_{14}b_{21}b_{32}b_{43}+b_{14}b_{21}b_{33}b_{42} \\ +\ b_{14}b_{22}b_{31}b_{43}+\ b_{14}b_{22}b_{33}b_{41}-b_{14}b_{23}b_{31}b_{42}+b_{14}b_{23}b_{32}b_{41} \end{bmatrix}}$$

$$\overline{N}_3=\frac{\begin{bmatrix} a_1[(b_{21}b_{32}b_{44}+b_{21}b_{34}b_{42})+(b_{22}b_{31}b_{44}+b_{22}b_{34}b_{41})+(b_{24}b_{31}b_{42}-b_{32}b_{24}b_{41})] \\ +a_2[(b_{11}b_{32}b_{44}+b_{11}b_{34}b_{42})+(b_{12}b_{31}b_{44}+b_{12}b_{34}b_{41})+(b_{14}b_{32}b_{41}-b_{14}b_{31}b_{42})] \\ +a_3[(b_{11}b_{22}b_{44}+b_{11}b_{24}b_{42})+(b_{12}b_{24}b_{41}-b_{12}b_{21}b_{44})+(b_{14}b_{21}b_{42}+b_{14}b_{22}b_{41})] \\ -a_4[(b_{11}b_{22}b_{34}+b_{11}b_{24}b_{32})+(b_{12}b_{21}b_{34}+b_{12}b_{24}b_{31})+(b_{14}b_{21}b_{32}+b_{14}b_{22}b_{31})] \end{bmatrix}}{\begin{bmatrix} b_{11}b_{22}b_{33}b_{44}-b_{11}b_{22}b_{34}b_{43}+\ b_{11}b_{23}b_{32}b_{44}+\ b_{11}b_{23}b_{34}b_{42}+b_{11}b_{24}b_{32}b_{43} \\ +b_{11}b_{24}b_{42}b_{33}-b_{12}b_{21}b_{33}b_{44}+\ b_{12}b_{21}b_{34}b_{43}+\ b_{12}b_{23}b_{31}b_{44}+b_{12}b_{23}b_{34}b_{41} \\ +b_{12}b_{24}b_{31}b_{43}+b_{12}b_{24}b_{33}b_{41}+b_{13}b_{21}b_{32}b_{44}+b_{13}b_{34}b_{42}b_{21}+b_{13}b_{22}b_{31}b_{44} \\ +b_{13}b_{22}b_{34}b_{41}+b_{13}b_{24}b_{31}b_{42}-b_{13}b_{24}b_{41}b_{32}+b_{14}b_{21}b_{32}b_{43}+b_{14}b_{21}b_{33}b_{42} \\ +\ b_{14}b_{22}b_{31}b_{43}+\ b_{14}b_{22}b_{33}b_{41}-b_{14}b_{23}b_{31}b_{42}+b_{14}b_{23}b_{32}b_{41} \end{bmatrix}}$$

$$\overline{N}_4 = \frac{\begin{bmatrix} a_1[(b_{21}b_{32}b_{43} + b_{21}b_{33}b_{42}) + (b_{22}b_{31}b_{43} + b_{22}b_{33}b_{41}) + (b_{23}b_{32}b_{41} - b_{23}b_{31}b_{42})] \\ +a_2[(b_{11}b_{32}b_{43} + b_{11}b_{33}b_{42}) + (b_{12}b_{31}b_{43} - b_{12}b_{33}b_{41}) + (b_{13}b_{31}b_{42} - b_{13}b_{32}b_{41})] \\ -a_3[(b_{11}b_{23}b_{42} - b_{11}b_{22}b_{43}) + (b_{12}b_{21}b_{43} + b_{12}b_{23}b_{41}) + (b_{13}b_{11}b_{22} - b_{13}b_{12}b_{21})] \\ +a_4[(b_{11}b_{22}b_{33} + b_{11}b_{23}b_{32}) + (b_{12}b_{23}b_{31} - b_{12}b_{21}b_{33}) + (b_{13}b_{21}b_{32} + b_{13}b_{22}b_{31})] \end{bmatrix}}{\begin{bmatrix} b_{11}b_{22}b_{33}b_{44} - b_{11}b_{22}b_{34}b_{43} + b_{11}b_{23}b_{32}b_{44} + b_{11}b_{23}b_{34}b_{42} + b_{11}b_{24}b_{32}b_{43} \\ +b_{11}b_{24}b_{42}b_{33} - b_{12}b_{21}b_{33}b_{44} + b_{12}b_{21}b_{34}b_{43} + b_{12}b_{23}b_{31}b_{44} + b_{12}b_{23}b_{34}b_{41} \\ +b_{12}b_{24}b_{31}b_{43} + b_{12}b_{24}b_{33}b_{41} + b_{13}b_{21}b_{32}b_{44} + b_{13}b_{34}b_{42}b_{21} + b_{13}b_{22}b_{31}b_{44} \\ +b_{13}b_{22}b_{34}b_{41} + b_{13}b_{24}b_{31}b_{42} - b_{13}b_{24}b_{41}b_{32} + b_{14}b_{21}b_{32}b_{43} + b_{14}b_{21}b_{33}b_{42} \\ + b_{14}b_{22}b_{31}b_{43} + b_{14}b_{22}b_{33}b_{41} - b_{14}b_{23}b_{31}b_{42} + b_{14}b_{23}b_{32}b_{41} \end{bmatrix}}$$

5. Positivity and Boundedness of the system

Positivity of the solutions $N_1(t), N_2(t), N_3(t), and N_4(t)$

Proposition: 5.1

Each solution of the system (1)-(4), accompanied by positive initial conditions, exists within the interval$[0,\infty)$, and all components, namely $N_1(t)$, $N_2(t)$, $N_3(t)$, and $N_4(t)$, remain non-negative for all $(t) \geq 0$.

Proof:

For $t\in [0,T]$, considering the continuity of the system (1)-(4), the solution $N_1(t)$, $N_2(t)$, $N_3(t)$ and $N_4(t)$with specified initial conditions exists uniquely on the interval $[0,T]$ where$0 < T < +\infty$.

Establishing the non-negativity of $N_1(t)$: the analytical solution for the density of the first population in the system (1) is expressed as:

$$N_1(t) = N_1(0)\left[\exp\int_0^T a_1 - b_{11}N_1(s) + b_{12}N_2(s) - b_{13}N_3(s) - b_{14}N_4(s)\right] ds$$

Given the inherent non-negativity of the exponential function and the assumed positivity of the initial population$N_1(0)$, it can be concluded that $N_1(t) > 0$ for all $t \geq 0$.

Establishing the non-negativity of $N_2(t)$: the analytical solution for the density of the second population in the system (2) is expressed as:

$$N_2(t) = N_2(0)\left[\exp\int_0^T a_2 - b_{22}N_2(s) + b_{21}N_1(s) - b_{23}N_3(s) - b_{24}N_4(s)\right] ds$$

Given the inherent non-negativity of the exponential function and the assumed positivity of the initial population$N_2(0)$, it can be concluded that $N_2(t) > 0$ for all $t \geq 0$.

Establishing the non-negativity of $N_3(t)$: the analytical solution for the density of the third population in the system (3) is expressed as:

$$N_3(t) = N_3(0)\left[\exp\int_0^T a_3 - b_{33}N_3(s) + b_{31}N_1(s) + b_{32}N_2(s) + b_{34}N_4(s)\right] ds$$

Given the inherent non-negativity of the exponential function and the assumed positivity of the initial population $N_3(0)$, it can be concluded that $N_3(t) > 0$ for all $t \geq 0$.

Establishing the non-negativity of $N_4(t)$: the analytical solution for the density of the fourth population in the system (4) is expressed as:

$$N_4(t) = N_4(0)\left[\exp\int_0^T a_4 - b_{44}N_4(s) + b_{41}N_1(s) + b_{42}N_2(s) + b_{43}N_3(s)\right] ds$$

Given the inherent non-negativity of the exponential function and the assumed positivity of the initial population $N_4(0)$, it can be concluded that $N_4(t) > 0$ for all $t \geq 0$.

Therefore, under the specified positive initial conditions, all solutions of the system (1)-(4) are positive for $t \geq 0$.

Boundedness of the solutions $\boldsymbol{N_1(t), N_2(t), N_3(t), \text{and} N_4(t)}$

Proposition: 5.2

Every solutions originating in the non-negative quadrantR_+^4 for the system (1)-(4) is characterized by being uniformly bounded.

Proof:

Let's establish the function u as the sum of $N_1(t)$, $N_2(t)$, $N_3(t)$, $N_4(t)$ denoted as

$$u = N_1(t) + N_2(t) + N_3(t) + N_4(t)$$

The derivative with respect to time yields

$$\frac{du}{dt}=\frac{dN_1}{dt}+\frac{dN_2}{dt}+\frac{dN_3}{dt}+\frac{dN_4}{dt}=\begin{bmatrix}(a_1N_1-b_{11}N_1^2+b_{12}N_1N_2-b_{13}N_1N_3-b_{14}N_1N_4)\\+(a_2N_2-b_{22}N_2^2+b_{21}N_1N_2-b_{23}N_2N_3-b_{24}N_2N_4)\\+(a_3N_3-b_{33}N_3^2+b_{31}N_1N_3+b_{32}N_2N_3+b_{34}N_3N_4)\\+(a_4N_4-b_{44}N_4^2+b_{41}N_1N_4+b_{42}N_2N_4+b_{43}N_3N_4)\end{bmatrix}$$

$$\text{Now, }\frac{du}{dt}+\delta u=\begin{bmatrix}(a_1N_1-b_{11}N_1^2)+(a_2N_2-b_{22}N_2^2)+(a_3N_3-b_{33}N_3^2)+(a_4N_4-b_{44}N_4^2)\\+\delta(N_1(t)+N_2(t)+N_3(t)+N_4(t))\end{bmatrix}$$

$$\frac{du}{dt}+\delta u=\left[\frac{(a_1+\delta)^2}{4b_{11}}+\frac{(a_2+\delta)^2}{4b_{22}}+\frac{(a_3+\delta)^2}{4b_{33}}+\frac{(a_4+\delta)^2}{4b_{44}}-b_{11}\left[N_1-\frac{(a_1+\delta)}{2b_{11}}\right]^2\right.$$
$$\left.-b_{22}\left[N_2-\frac{(a_2+\delta)}{2b_{22}}\right]^2-b_{33}\left[N_3-\frac{(a_3+\delta)}{2b_{33}}\right]^2-b_{44}\left[N_4-\frac{(a_4+\delta)}{2b_{44}}\right]^2\right]$$

$$\frac{du}{dt}+\delta u=\left[\frac{(a_1+\delta)^2}{4b_{11}}+\frac{(a_2+\delta)^2}{4b_{22}}+\frac{(a_3+\delta)^2}{4b_{33}}+\frac{(a_4+\delta)^2}{4b_{44}}\right]$$

$$\frac{du}{dt}+\delta u\le K\ (\text{say})$$

Where $K=\frac{(a_1+\delta)^2}{4b_{11}}+\frac{(a_2+\delta)^2}{4b_{22}}+\frac{(a_3+\delta)^2}{4b_{33}}+\frac{(a_4+\delta)^2}{4b_{44}}$ which forms a linear differential equation in u. upon solving we obtain, $u\le\frac{K}{\delta}+Ce^{-\delta t}$ where C is an integrating constant. Given $t=0$, implies $u=0$, we determine $C=-\frac{K}{\delta}$. Consequently, we get $u\le\frac{K}{\delta}(1-e^{-\delta t})$ and since $u\ge 0$ it follows that $0\le u\le\frac{K}{\delta}(1-e^{-\delta t})$. This implies that all solutions of the system are bounded.

Thus, the solutions to the system (1)-(4) conform within the bounded region ω where

$$\omega=\left\{(N_1(t),N_2(t),N_3(t),N_4(t))\epsilon R_+^4:\ 0\le u\le\frac{K}{\delta}(1-e^{-\delta t})+\varepsilon, for any \varepsilon>0\right\}$$

This demonstrate that all species are uniformly constrained for any initial value in R_+^4.

Based on the earlier results, we assert the presence of positive values for $(\beta_1,\beta_2,\beta_3,\beta_4)$ such that

$$u(t)\subset R_+^4=\{(N_1(t),N_2(t),N_3(t),N_4(t)):0\le N_1(t)\le\beta_1,0\le N_2(t)\le\beta_2,0\le N_3(t)\le\beta_3,0\le N_4(t)\le\beta_4,\}$$

Hence, the solutions to the system (1) -(4) are proven to be uniformly bounded over time within its environment. This concludes the proof.

6. Local stability of the positive equilibrium state

This section uses the variational (jacobian) matrix to examine the stability of the equilibrium for each condition where a species is present or extinct in the ecological system. The output of the matrices, obtained using the characteristic roots approach and the Routh-Hurwitz criterion, displays the stability performance at each state. The variational matrix which is given by

$$V=\begin{bmatrix}A_{11}&b_{12}N_1&-b_{13}N_1&-b_{14}N_1\\b_{21}N_2&A_{22}&-b_{23}N_2&-b_{24}N_2\\b_{31}N_3&b_{32}N_3&A_{33}&b_{34}N_3\\b_{41}N_4&b_{42}N_4&b_{43}N_4&A_{44}\end{bmatrix}\ldots(5)$$

Where $A_{11}=a_1-2b_{11}N_1+b_{12}N_2-b_{13}N_3-b_{14}N_4$, $A_{22}=a_2-2b_{22}N_2+b_{21}N_1-b_{23}N_3-b_{24}N_4$, $A_{33}=a_3-2b_{33}N_3+b_{31}N_1+b_{32}N_2+b_{34}N_4$, $A_{44}=a_4-2b_{44}N_4+b_{41}N_1+b_{42}N_2+b_{43}N_3$

Theorem: 6.1

The local steady state of the dynamic system (5) is unstable for the following states of equilibrium

(i)Trivial equilibrium point $E_T(0,0,0,\ 0)$.

(ii)Axial equilibrium point

(a) $E_{A1}\left(\frac{a_1}{b_{11}},0,0,0\right)$, (b) $E_{A2}\left(0,\frac{a_2}{b_{22}},0,0\right)$, (c) $E_{A3}\left(0,0,\frac{a_3}{b_{33}},0\right)$,(d) $E_{A4}\left(0,0,0,\ \frac{a_4}{b_{44}}\right)$.

Proof:

(i)The variational matrix of the system at the trivial equilibrium point $E_T(0,0,0,\ 0)$produces positive eigen values, denoted as $\lambda=a_1,a_2,a_3$ and a_4 which signify an unstable state due to the consistently positive birth rate of the species. Thus, the equilibrium points E_T in nature exhibit instability in their state of stability.

(ii) (a)The eigen values of the variational matrix system at the axial point $E_{A1}\left(\frac{a_1}{b_{11}},0,0,0\right)$for the first prey are given by $\lambda = -a_1,\ a_2 + b_{21}\frac{a_1}{b_{11}},\ a_3 + b_{31}\frac{a_1}{b_{11}}$ and $a_4 + b_{41}\frac{a_1}{b_{11}}$. Here the first eigen value is negative and other three eigen values are positive. Thus the positive nature of these eigen values arises from the species' birth rate and interactions with other species. Consequently, the equilibrium points E_{A1} indicate an unstable state of steadiness and saddle point exists.

(b)The variational matrix of the system at the axial point $E_{A2}\left(0,\frac{a_2}{b_{22}},0,0\right)$for the second prey provides the eigen values, represented as $\lambda = a_1 + b_{12}\frac{a_2}{b_{22}}, -a_2,\ a_3 + b_{32}\frac{a_2}{b_{22}}$, and $a_4 + b_{42}\frac{a_2}{b_{22}}$,. Here the second eigen value is negative and other threes are positive. Thus the positivity is attributed to the birth rate of the species and the rate of interaction between the species. As a result, the state of steadiness is unstable and saddle point exists at the equilibrium points E_{A2}.

(c)The eigen values resulting from the variational matrix of the system at the axial point $E_{A3}\left(0,0,\frac{a_3}{b_{33}},0\right)$of the primary predator are given by $\lambda_1 = a_1 - b_{13}\frac{a_3}{b_{33}},\ \lambda_2 = a_2 - b_{23}\frac{a_3}{b_{33}},\ \lambda_3 = -a_3,\ \lambda_4 = a_4 + b_{43}\frac{a_3}{b_{33}}$.Here the third eigen value is negative. Then the Stability condition of the above eigen values as follows,

(i) The system is stable in $N_1N_2N_3$ plane if $a_1 < b_{13}\frac{a_3}{b_{33}},\ a_2 < b_{23}\frac{a_3}{b_{33}}$then$\lambda_1 < 0,\ \lambda_2 < 0, and\ \lambda_3 < 0$.

(ii) The system is unstable in N_1N_2 Plane if $a_1 > b_{13}\frac{a_3}{b_{33}},\ a_2 > b_{23}\frac{a_3}{b_{33}}$then$\lambda_1 > 0,\ \lambda_2 > 0$.

But it observes that$\lambda_4 > 0$ and $\lambda_3 < 0$. As a consequence, the equilibrium points at E_{A3} exhibit unstable and saddle point exists in terms of steadiness.

(d) The system of the variational matrix at axial point of secondary predator $E_{A4}\left(0,0,0,\frac{a_4}{b_{44}}\right)$ gives the eigen values $\lambda_1 = a_1 - b_{14}\frac{a_4}{b_{44}},\ \lambda_2 = a_2 - b_{24}\frac{a_4}{b_{44}},\ \lambda_3 = a_3 + b_{34}\frac{a_4}{b_{44}},\ \lambda_4 = -a_4$ Here the third eigen value is negative. Then the Stability condition of the above eigen values as follows,

(i) The system is stable in $N_1N_2N_4$ plane if $a_1 < b_{14}\frac{a_4}{b_{44}},\ a_2 < b_{24}\frac{a_4}{b_{44}}$then $\lambda_1 < 0,\ \lambda_2 < 0, and\ \lambda_4 < 0$.

(ii) The system is unstable in N_1N_2 Plane if $a_1 > b_{14}\frac{a_4}{b_{44}},\ a_2 > b_{24}\frac{a_4}{b_{44}}$then$\lambda_1 > 0,\ \lambda_2 > 0$.

But it observes that $\lambda_3 > 0\ and \lambda_4 < 0$. Hence the state of steadiness is unstable and saddle point exist to the equilibrium points E_{A4}.

Theorem: 6.2

The local steady state of the dynamic system (5) shows the stability condition for the following semi-interior equilibrium point when two of the species extinct state.

(a) Equilibrium Point $E_{S1}(\overline{N}_1, \overline{N}_2, 0, 0)$ is unstable and saddle point exist,

if$b_{11}b_{22} > b_{12}b_{21}$ or $b_{11}b_{22} < b_{12}b_{21}$and $\begin{bmatrix}(a_1b_{11}b_{22} + a_2b_{11}b_{12})(a_2b_{11}b_{22} + a_1b_{21}b_{22}) \\ > (a_1b_{12}b_{22} + a_2{b_{12}}^2)(a_2b_{11}b_{21} + a_1{b_{21}}^2)\end{bmatrix}$.

(b) Equilibrium Point $E_{S2}(0, \overline{N}_2, \overline{N}_3, 0)$ is unstable and saddle point exist,

if $\lambda_2 < 0, \lambda_3 < 0$ and $(\lambda_1 < 0\ or \lambda_1 > 0),\ \lambda_4 > 0$.

(c) Equilibrium Point $E_{S3}(0,0,\overline{N}_3,\overline{N}_4)$is asymptotically stable if$(b_{33}b_{44} > b_{34}b_{43})$and $\begin{bmatrix}(a_3b_{33}b_{44} + a_4b_{33}b_{34})(a_3b_{43}b_{44} - a_4b_{33}b_{44}) \\ > (a_3b_{34}b_{44} + a_4{b_{34}}^2)(a_4b_{33}b_{43} + a_3{b_{43}}^2)\end{bmatrix}$. Otherwise it is unstable.

(d) Equilibrium state$E_{S4}(\overline{N}_1, 0,0, \overline{N}_4)$ is unstable and saddle point exist, if $\lambda_1 < 0, \lambda_4 < 0$ and $(\lambda_2 < 0\ or \lambda_2 > 0)\ , \lambda_3 > 0$ (always).

(e) Equilibrium Point $E_{S5}(\overline{N}_1, 0, \overline{N}_3, 0)$ is unstable and saddle point exist, if $\lambda_1 < 0, \lambda_3 < 0$ and $(\lambda_2 < 0\ or\ \lambda_2 > 0), \lambda_4 > 0$ (always).

(f) Equilibrium Point $E_{S6}(0, \overline{N}_2, 0, \overline{N}_4)$ is unstable and saddle point exist, if$\lambda_2 < 0\ ,\ \lambda_4 < 0$and

$$(\lambda_1 < 0\ or\ \lambda_1 > 0), \lambda_3 > 0.$$

Proof:

(a) The system of the variational matrix at predator's free semi-interior equilibrium point $E_{S1}(\overline{N}_1, \overline{N}_2, 0, 0)$in matrix (5) gives the eigen values

$$\lambda_1 = \frac{1}{2}\left[\begin{array}{c} -\left(\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12}) + (a_2b_{11}b_{22} + a_1b_{21}b_{22})}{b_{11}b_{22} - b_{12}b_{21}}\right)\right) \\ + \sqrt{\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12}) + (a_2b_{11}b_{22} + a_1b_{21}b_{22})}{b_{11}b_{22} - b_{12}b_{21}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12})(a_2b_{11}b_{22} + a_1b_{21}b_{22}) - (a_1b_{12}b_{22} + a_2{b_{12}}^2)(a_2b_{11}b_{21} + a_1{b_{21}}^2)}{(b_{11}b_{22} - b_{12}b_{21})^2}\right)} \end{array}\right],$$

$$\lambda_2 = \frac{1}{2}\left[\begin{array}{c} -\left(\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12}) + (a_2b_{11}b_{22} + a_1b_{21}b_{22})}{b_{11}b_{22} - b_{12}b_{21}}\right)\right) \\ - \sqrt{\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12}) + (a_2b_{11}b_{22} + a_1b_{21}b_{22})}{b_{11}b_{22} - b_{12}b_{21}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{22} + a_2b_{11}b_{12})(a_2b_{11}b_{22} + a_1b_{21}b_{22}) - (a_1b_{12}b_{22} + a_2{b_{12}}^2)(a_2b_{11}b_{21} + a_1{b_{21}}^2)}{(b_{11}b_{22} - b_{12}b_{21})^2}\right)} \end{array}\right]$$

$$\lambda_3 = \left(\frac{a_3b_{11}b_{22} - a_3b_{12}b_{21} + a_1b_{22}b_{31} + a_2b_{12}b_{31} + a_2b_{11}b_{32} + a_1b_{21}b_{32}}{b_{11}b_{22} - b_{12}b_{21}}\right) \text{ and}$$

$$\lambda_4 = \left(\frac{a_4b_{11}b_{22} - a_4b_{12}b_{21} + a_1b_{22}b_{41} + a_2b_{12}b_{41} + a_2b_{11}b_{42} + a_1b_{21}b_{42}}{b_{11}b_{22} - b_{12}b_{21}}\right)$$

If $b_{11}b_{22} > b_{12}b_{21}$, then the sum of the roots $\left(\frac{(a_1b_{11}b_{22}+a_2b_{11}b_{12})+(a_2b_{11}b_{22}+a_1b_{21}b_{22})}{b_{11}b_{22}-b_{12}b_{21}}\right)$ is negativeand the product of the roots$\left(\frac{(a_1b_{11}b_{22}+a_2b_{11}b_{12})(a_2b_{11}b_{22}+a_1b_{21}b_{22})-(a_1b_{12}b_{22}+a_2{b_{12}}^2)(a_2b_{11}b_{21}+a_1{b_{21}}^2)}{(b_{11}b_{22}-b_{12}b_{21})^2}\right)$is positivefrom the first two eigen values. Then the roots of the above equation are real and negative or complex having negative real part.So that the state will be asymptotically stable in N_1N_2plane. Butit is clear that

$$\lambda_3 = \left(\frac{a_3b_{11}b_{22} - a_3b_{12}b_{21} + a_1b_{22}b_{31} + a_2b_{12}b_{31} + a_2b_{11}b_{32} + a_1b_{21}b_{32}}{b_{11}b_{22} - b_{12}b_{21}}\right) \textit{and}$$

$$\lambda_4 = \left(\frac{a_4b_{11}b_{22} - a_4b_{12}b_{21} + a_1b_{22}b_{41} + a_2b_{12}b_{41} + a_2b_{11}b_{42} + a_1b_{21}b_{42}}{b_{11}b_{22} - b_{12}b_{21}}\right).$$

are positive sign whereas $b_{11}b_{22} > b_{12}b_{21}$ or negative sign whereas $b_{11}b_{22} < b_{12}b_{21}$.
The stability condition is hold based on the following two cases,

Case(i): If $b_{11}b_{22} > b_{12}b_{21}$and $\left[\begin{array}{c}(a_1b_{11}b_{22} + a_2b_{11}b_{12})(a_2b_{11}b_{22} + a_1b_{21}b_{22}) > \\ (a_1b_{12}b_{22} + a_2{b_{12}}^2)(a_2b_{11}b_{21} + a_1{b_{21}}^2)\end{array}\right]$
then $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_3 > 0, \lambda_4 > 0$.

Case(ii): If $b_{11}b_{22} < b_{12}b_{21}$ and $\left[\begin{array}{c}(a_1b_{11}b_{22} + a_2b_{11}b_{12})(a_2b_{11}b_{22} + a_1b_{21}b_{22}) > \\ (a_1b_{12}b_{22} + a_2{b_{12}}^2)(a_2b_{11}b_{21} + a_1{b_{21}}^2)\end{array}\right]$
then $\lambda_1 > 0, \lambda_2 > 0and\lambda_3 < 0, \lambda_4 < 0$.

Hence the system of predator's free state $V(E_1)(\overline{N}_1, \overline{N}_2, 0, 0)$ is unstable and saddle points exists at the coexistence state at both cases.

(b) The eigen values of the variational matrix are obtained at the semi-interior equilibrium point E_{S2} $(0, \overline{N}_2, \overline{N}_3, 0)$ in matrix (5), where the second prey and primary predator coexist.

$$\lambda_2 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_2b_{22}b_{33} - a_3b_{22}b_{23} + a_3b_{22}b_{33}+a_2b_{32}b_{33}}{b_{22}b_{33} + b_{23}b_{32}}\right) \\ + \sqrt{\left(\frac{a_3b_{22}b_{23} - a_2b_{22}b_{33} - a_3b_{22}b_{33}-a_2b_{32}b_{33}}{b_{22}b_{33} + b_{23}b_{32}}\right)^2 - 4\left(\frac{(a_2b_{22}b_{33} - a_3b_{22}b_{23})(a_3b_{22}b_{33}+a_2b_{32}b_{33}) + (a_3{b_{23}}^2-a_2b_{23}b_{33})(a_3b_{22}b_{32} + a_2{b_{32}}^2)}{(b_{22}b_{33} + b_{23}b_{32})^2}\right)} \end{array}\right],$$

$$\lambda_3 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_2b_{22}b_{33} - a_3b_{22}b_{23} + a_3b_{22}b_{33}+a_2b_{32}b_{33}}{b_{22}b_{33} + b_{23}b_{32}}\right) \\ - \sqrt{\left(\frac{a_3b_{22}b_{23} - a_2b_{22}b_{33} - a_3b_{22}b_{33}-a_2b_{32}b_{33}}{b_{22}b_{33} + b_{23}b_{32}}\right)^2 - 4\left(\frac{(a_2b_{22}b_{33} - a_3b_{22}b_{23})(a_3b_{22}b_{33}+a_2b_{32}b_{33}) + (a_3{b_{23}}^2-a_2b_{23}b_{33})(a_3b_{22}b_{32} + a_2{b_{32}}^2)}{(b_{22}b_{33} + b_{23}b_{32})^2}\right)} \end{array}\right],$$

$$\lambda_1 = \left(\frac{a_1b_{22}b_{33} + a_1b_{23}b_{32} + a_2b_{12}b_{33} - a_3b_{12}b_{23} - a_3b_{13}b_{22} - a_2b_{13}b_{32}}{b_{22}b_{33} + b_{23}b_{32}}\right) \textit{ and}$$

$$\lambda_4 = \left(\frac{a_4b_{22}b_{33} + a_4b_{23}b_{32} + a_2b_{33}b_{42} - a_3b_{23}b_{42} + a_3b_{22}b_{43} + a_2b_{32}b_{43}}{b_{22}b_{33} + b_{23}b_{32}}\right) > 0$$

If the sum of the roots $\left(\frac{a_2b_{22}b_{33}-a_3b_{22}b_{23}+a_3b_{22}b_{33}+a_2b_{32}b_{33}}{b_{22}b_{33}+b_{23}b_{32}}\right)$ is negative and the product of the roots $\left(\frac{(a_2b_{22}b_{33}-a_3b_{22}b_{23})(a_3b_{22}b_{33}+a_2b_{32}b_{33})+(a_3{b_{23}}^2-a_2b_{23}b_{33})(a_3b_{22}b_{32}+a_2{b_{32}}^2)}{(b_{22}b_{33}+b_{23}b_{32})^2}\right)$ is positive from the second and third eigen values. Then the roots of the above equation are real and negative or complex having negative real part. So, the state will be asymptotically stable in N_2N_3 plane. But it is clear that

$\lambda_1 > 0$, When the condition exists for $(a_1b_{22}b_{33} + a_1b_{23}b_{32} + a_2b_{12}b_{33}) > (a_3b_{12}b_{23} + a_3b_{13}b_{22} + a_2b_{13}b_{32})$
$\lambda_1 < 0$, When the condition exists for $(a_1b_{22}b_{33} + a_1b_{23}b_{32} + a_2b_{12}b_{33}) < (a_3b_{12}b_{23} + a_3b_{13}b_{22} + a_2b_{13}b_{32})$
and $\lambda_4 > 0$
Therefore, $\lambda_2 < 0$, $\lambda_3 < 0$ and $\lambda_1 < 0$ *or* $\lambda_1 > 0$, $\lambda_4 > 0$ (always). Hence the system of second prey and the primary predator state exists of $V(E_{S2})$ $(0,\overline{N}_2,\overline{N}_3,0)$ is unstable and saddle points exists.
(C) The system of the variational matrix yields the eigen values at the semi-interior equilibrium point $E_{S3}(0,0,\overline{N}_3,\overline{N}_4)$ in matrix (5) where the primary and secondary predators coexist.

$$\lambda_1 = \left(\frac{a_1b_{33}b_{44} - a_1b_{34}b_{43} - a_3b_{13}b_{44} - a_4b_{13}b_{34} - a_4b_{14}b_{33} - a_3b_{14}b_{43}}{b_{33}b_{44} - b_{34}b_{43}}\right) < 0, \quad \text{If } b_{33}b_{44} > b_{34}b_{43}$$

$$\lambda_2 = \left(\frac{a_2b_{33}b_{44} - a_2b_{34}b_{43} - a_3b_{23}b_{44} - a_4b_{23}b_{34} - a_4b_{24}b_{33} - a_3b_{24}b_{43}}{b_{33}b_{44} - b_{34}b_{43}}\right) < 0. \text{If } b_{33}b_{44} > b_{34}b_{43}$$

$$\lambda_3 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_3b_{33}b_{44} + a_4b_{33}b_{34} - a_4b_{33}b_{44} + a_3b_{43}b_{44}}{b_{33}b_{44} - b_{34}b_{43}}\right) \\ + \sqrt{\left(\frac{a_3b_{33}b_{44} + a_4b_{33}b_{34} - a_4b_{33}b_{44} + a_3b_{43}b_{44}}{b_{33}b_{44} - b_{34}b_{43}}\right)^2 - 4\left(\frac{(a_3b_{33}b_{44} + a_4b_{33}b_{34})(a_3b_{43}b_{44} - a_4b_{33}b_{44}) - (a_3b_{34}b_{44} + a_4b_{34}^2)(a_4b_{33}b_{43} + a_3b_{43}^2)}{(b_{33}b_{44} - b_{34}b_{43})^2}\right)} \end{array}\right]$$

and

$$\lambda_4 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_3b_{33}b_{44} + a_4b_{33}b_{34} - a_4b_{33}b_{44} + a_3b_{43}b_{44}}{b_{33}b_{44} - b_{34}b_{43}}\right) \\ - \sqrt{\left(\frac{a_3b_{33}b_{44} + a_4b_{33}b_{34} - a_4b_{33}b_{44} + a_3b_{43}b_{44}}{b_{33}b_{44} - b_{34}b_{43}}\right)^2 - 4\left(\frac{(a_3b_{33}b_{44} + a_4b_{33}b_{34})(a_3b_{43}b_{44} - a_4b_{33}b_{44}) - (a_3b_{34}b_{44} + a_4b_{34}^2)(a_4b_{33}b_{43} + a_3b_{43}^2)}{(b_{33}b_{44} - b_{34}b_{43})^2}\right)} \end{array}\right]$$

The stability condition is based on following two condition hold
Case (i) If $b_{33}b_{44} > b_{34}b_{43}$ and $\left[\begin{array}{c}(a_3b_{33}b_{44} + a_4b_{33}b_{34})(a_3b_{43}b_{44} - a_4b_{33}b_{44}) \\ > (a_3b_{34}b_{44} + a_4b_{34}^2)(a_4b_{33}b_{43} + a_3b_{43}^2)\end{array}\right]$ then $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$, $\lambda_4 < 0$. Hencethese eigen values are produced negative sign then the system of $V(E_{S3})$ $(0, 0, \overline{N}_3, \overline{N}_4)$ is asymptotically stable.
Case(ii) If $b_{33}b_{44} < b_{34}b_{43}$ and $\left[\begin{array}{c}(a_3b_{33}b_{44} + a_4b_{33}b_{34})(a_3b_{43}b_{44} - a_4b_{33}b_{44}) \\ > (a_3b_{34}b_{44} + a_4b_{34}^2)(a_4b_{33}b_{43} + a_3b_{43}^2)\end{array}\right]$ then $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$, $\lambda_4 > 0$. Hence these eigen values are produced opposite sign then the system of the state $V(E_{S3})(0, 0, \overline{N}_3, \overline{N}_4)$ is unstable.

(d) The eigen values of the variational matrix are obtained at the semi-interior equilibrium point $E_{S4}(\overline{N}_1, 0,0, \overline{N}_4)$ in matrix (5), where The first prey and secondary predator coexist.

$$\lambda_2 = \left(\frac{a_2b_{11}b_{44} + a_2b_{14}b_{41} + a_1b_{21}b_{44} - a_4b_{14}b_{21} - a_4b_{11}b_{24} - a_1b_{24}b_{41}}{b_{11}b_{44} + b_{14}b_{41}}\right)$$

$$\lambda_3 = \left(\frac{a_3b_{11}b_{44} + a_3b_{14}b_{41} + a_1b_{31}b_{44} - a_4b_{31}b_{14} + a_4b_{11}b_{34} + a_1b_{34}b_{41}}{b_{11}b_{44} + b_{14}b_{41}}\right) > 0$$

$$\lambda_1 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_1b_{11}b_{44} - a_4b_{11}b_{14} + a_4b_{11}b_{44} + a_1b_{41}b_{44}}{b_{11}b_{44} + b_{14}b_{41}}\right) \\ + \sqrt{\left(\frac{a_1b_{11}b_{44} - a_4b_{11}b_{14} + a_4b_{11}b_{44} + a_1b_{41}b_{44}}{b_{11}b_{44} + b_{14}b_{41}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{44} - a_4b_{11}b_{14})(a_4b_{11}b_{44} + a_1b_{41}b_{44}) + (a_4b_{14}^2 - a_1b_{14}b_{44})(a_4b_{11}b_{41} + a_1b_{41}^2)}{(b_{11}b_{44} + b_{14}b_{41})^2}\right)} \end{array}\right]$$

and

$$\lambda_4 = \frac{1}{2}\left[\begin{array}{c} -\left(\frac{a_1b_{11}b_{44} - a_4b_{11}b_{14} + a_4b_{11}b_{44} + a_1b_{41}b_{44}}{b_{11}b_{44} + b_{14}b_{41}}\right) \\ - \sqrt{\left(\frac{a_1b_{11}b_{44} - a_4b_{11}b_{14} + a_4b_{11}b_{44} + a_1b_{41}b_{44}}{b_{11}b_{44} + b_{14}b_{41}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{44} - a_4b_{11}b_{14})(a_4b_{11}b_{44} + a_1b_{41}b_{44}) + (a_4b_{14}^2 - a_1b_{14}b_{44})(a_4b_{11}b_{41} + a_1b_{41}^2)}{(b_{11}b_{44} + b_{14}b_{41})^2}\right)} \end{array}\right]$$

If $[(a_1b_{11}b_{44} - a_4b_{11}b_{14})(a_4b_{11}b_{44} + a_1b_{41}b_{44}) + (a_4b_{14}^2 - a_1b_{14}b_{44})(a_4b_{11}b_{41} + a_1b_{41}^2)] > 0$, then the eigen values are produced negative ie., $\lambda_1 < 0$, $\lambda_4 < 0$.So, the state will be asymptotically stable in N_1N_4 plane.
But it's observed that
$\lambda_2 > 0$, when the condition exists for $(a_2b_{11}b_{44} + a_2b_{14}b_{41} + a_1b_{21}b_{44}) > (a_4b_{14}b_{21} + a_4b_{11}b_{24} + a_1b_{24}b_{41})$.

$\lambda_2 < 0$, when the condition exists for $(a_2b_{11}b_{44} + a_2b_{14}b_{41} + a_1b_{21}b_{44}) < (a_4b_{14}b_{21} + a_4b_{11}b_{24} + a_1b_{24}b_{41})$
and$\lambda_3 > 0$ is always positive
Hence, $\lambda_1 < 0, \lambda_4 < 0$ and $\lambda_2 < 0$ *or* $\lambda_2 > 0$, $\lambda_3 > 0$(always). Consequently, the system of the state $E_{S4}(\overline{N}_1, 0, 0, \overline{N}_4)$ is unstable and saddle point exists.
(e)The variational matrix of the system at the semi-interior equilibrium point$E_{S5}(\overline{N}_1, 0, \overline{N}_3, 0)$ for the coexistence of the first prey and primary predator produces the eigen values,

$$\lambda_2 = \left(\frac{a_2b_{11}b_{33} + a_2b_{13}b_{31} + a_1b_{21}b_{33} - a_3b_{21}b_{13} - a_3b_{11}b_{23} - a_1b_{31}b_{23}}{b_{11}b_{33} + b_{13}b_{31}}\right)$$

$$\lambda_4 = \left(\frac{a_4b_{11}b_{33} + a_4b_{13}b_{31} + a_1b_{33}b_{41} - a_3b_{13}b_{41} + a_3b_{11}b_{43} + a_1b_{31}b_{43}}{b_{11}b_{33} + b_{13}b_{31}}\right) > 0$$

$$\lambda_1 = \frac{1}{2}\left[-\left(\frac{a_1b_{11}b_{33} - a_3b_{11}b_{13} + a_3b_{11}b_{33} + a_1b_{31}b_{33}}{b_{11}b_{33} + b_{13}b_{31}}\right) + \sqrt{\left(\frac{a_1b_{11}b_{33} - a_3b_{11}b_{13} + a_3b_{11}b_{33} + a_1b_{31}b_{33}}{b_{11}b_{33} + b_{13}b_{31}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{33} - a_3b_{11}b_{13})(a_3b_{11}b_{33} + a_1b_{31}b_{33}) + \left(a_1b_{13}b_{33} - a_3{b_{13}}^2\right)\left(a_3b_{11}b_{31} + a_1{b_{31}}^2\right)}{(b_{11}b_{33} + b_{13}b_{31})^2}\right)}\right]$$

and

$$\lambda_3 = \frac{1}{2}\left[-\left(\frac{a_1b_{11}b_{33} - a_3b_{11}b_{13} + a_3b_{11}b_{33} + a_1b_{31}b_{33}}{b_{11}b_{33} + b_{13}b_{31}}\right) - \sqrt{\left(\frac{a_1b_{11}b_{33} - a_3b_{11}b_{13} + a_3b_{11}b_{33} + a_1b_{31}b_{33}}{b_{11}b_{33} + b_{13}b_{31}}\right)^2 - 4\left(\frac{(a_1b_{11}b_{33} - a_3b_{11}b_{13})(a_3b_{11}b_{33} + a_1b_{31}b_{33}) + \left(a_1b_{13}b_{33} - a_3{b_{13}}^2\right)\left(a_3b_{11}b_{31} + a_1{b_{31}}^2\right)}{(b_{11}b_{33} + b_{13}b_{31})^2}\right)}\right]$$

If $(a_1b_{11}b_{33} - a_3b_{11}b_{13})(a_3b_{11}b_{33} + a_1b_{31}b_{33}) + \left(a_1b_{13}b_{33} - a_3{b_{13}}^2\right)\left(a_3b_{11}b_{31} + a_1{b_{31}}^2\right) > 0$, then the eigen values are produced negative ie., $\lambda_1 < 0$, $\lambda_3 < 0$.So that, the state will be asymptotically stable in N_1N_3 plane.
But it's observed that
$\lambda_2 > 0$, When the condition exists for $(a_2b_{11}b_{33} + a_2b_{13}b_{31} + a_1b_{21}b_{33}) > (a_3b_{21}b_{13} + a_3b_{11}b_{23} + a_1b_{31}b_{23})$.
$\lambda_2 < 0$, When the condition exists for $(a_2b_{11}b_{33} + a_2b_{13}b_{31} + a_1b_{21}b_{33}) < (a_3b_{21}b_{13} + a_3b_{11}b_{23} + a_1b_{31}b_{23})$
and $\lambda_4 > 0$ is always positive.
Hence $\lambda_1 < 0$, $\lambda_3 < 0$ and $\lambda_2 < 0$ *or* $\lambda_2 > 0, \lambda_4 > 0$ (always).Therefore, the system of the state$E_{S5}(\overline{N}_1, 0, \overline{N}_3, 0)$is unstable and saddle point exists.
(f) The eigen values of the variational matrix are obtained at the semi-interior equilibrium point E_{S6} $(0, \overline{N}_2, 0, \overline{N}_4)$ in matrix (5), where the second prey and secondary predator coexist.

$$\lambda_1 = \left(\frac{a_1b_{22}b_{44} + a_1b_{24}b_{42} + a_2b_{12}b_{44} - a_4b_{12}b_{24} - a_4b_{14}b_{22} - a_2b_{14}b_{42}}{b_{22}b_{44} + b_{24}b_{42}}\right),$$

$$\lambda_3 = \left(\frac{a_3b_{22}b_{44} + a_3\,b_{24}b_{42} + a_4b_{22}b_{34} + a_2b_{34}b_{42} + a_2b_{32}b_{44} - a_4b_{24}b_{32}}{b_{22}b_{44} + b_{24}b_{42}}\right) > 0,$$

$$\lambda_2 = \frac{1}{2}\left[-\left(\frac{a_2b_{22}b_{44} - a_4b_{22}b_{24} + a_4b_{22}b_{44} + a_2b_{42}b_{44}}{b_{22}b_{44} + b_{24}b_{42}}\right) + \sqrt{\left(\frac{a_2b_{22}b_{44} - a_4b_{22}b_{24} + a_4b_{22}b_{44} + a_2b_{42}b_{44}}{b_{22}b_{44} + b_{24}b_{42}}\right)^2 - 4\left(\frac{(a_2b_{22}b_{44} - a_4b_{22}b_{24})(a_4b_{22}b_{44} + a_2b_{42}b_{44}) + \left(a_2b_{24}b_{44} - a_4{b_{24}}^2\right)\left(a_4b_{22}b_{42} + a_2{b_{42}}^2\right)}{(b_{22}b_{44} + b_{24}b_{42})^2}\right)}\right]$$

and

$$\lambda_4 = \frac{1}{2}\left[-\left(\frac{a_2b_{22}b_{44} - a_4b_{22}b_{24} + a_4b_{22}b_{44} + a_2b_{42}b_{44}}{b_{22}b_{44} + b_{24}b_{42}}\right) - \sqrt{\left(\frac{a_2b_{22}b_{44} - a_4b_{22}b_{24} + a_4b_{22}b_{44} + a_2b_{42}b_{44}}{b_{22}b_{44} + b_{24}b_{42}}\right)^2 - 4\left(\frac{(a_2b_{22}b_{44} - a_4b_{22}b_{24})(a_4b_{22}b_{44} + a_2b_{42}b_{44}) + \left(a_2b_{24}b_{44} - a_4{b_{24}}^2\right)\left(a_4b_{22}b_{42} + a_2{b_{42}}^2\right)}{(b_{22}b_{44} + b_{24}b_{42})^2}\right)}\right]$$

If $(a_2b_{22}b_{44} - a_4b_{22}b_{24})(a_4b_{22}b_{44} + a_2b_{42}b_{44}) + \left(a_2b_{24}b_{44} - a_4{b_{24}}^2\right)\left(a_4b_{22}b_{42} + a_2{b_{42}}^2\right) > 0$, then the eigen values are produced negative sign ie., $\lambda_2 < 0$, $\lambda_4 < 0$. So, the state will be asymptotically stable in N_2N_4 plane.
But it's observed that

$\lambda_1 > 0$, when the condition exist for$(a_1b_{22}b_{44} + a_1b_{24}b_{42} + a_2b_{12}b_{44}) > (a_4b_{12}b_{24} + a_4b_{14}b_{22} + a_2b_{14}b_{42})$
$\lambda_1 < 0$, when the condition exist for$(a_1b_{22}b_{44} + a_1b_{24}b_{42} + a_2b_{12}b_{44}) < (a_4b_{12}b_{24} + a_4b_{14}b_{22} + a_2b_{14}b_{42})$
and $\lambda_3 > 0$ is positive.
Therefore, $\lambda_2 < 0$, $\lambda_4 < 0$and $(\lambda_1 < 0 \; or \; \lambda_1 > 0), \lambda_3 > 0$.Therefore, the system of the state $E_{S6}(0, \overline{N}_2, 0, \overline{N}_4)$ is unstable and saddle point exists.

Theorem: 6.3
The local steady state of dynamic system (5) shows the stability condition when one of the species reaches extinction at the specified semi-interior equilibrium point.
(g) The equilibrium point $E_{S7}(\overline{N}_1, \overline{N}_2, \overline{N}_3, 0)$ is unstable for the reason that $\lambda_4 > 0$.
(h) The equilibrium point $E_{S8}(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ is asymptotically stable if $(a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4) < 0$ and $q_1 > 0, q_2 > 0, q_3 > 0$, $q_1q_2 > 0$ and $(q_1q_2 - q_3) > 0$ are satisfied. otherwise it is unstable.
(i) The equilibrium point $E_{S9}(\overline{N}_1, 0, \overline{N}_3, \overline{N}_4)$ is asymptotically stable if $(a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4) < 0$ and
$r_1 > 0, r_2 > 0, \; r_3 > 0, \; r_1r_2 > 0$ and $(r_1r_2 - r_3) > 0$ are satisfied. Otherwise it is unstable.
(j) The equilibrium point $E_{S10}(\overline{N}_1, \overline{N}_2, 0, \overline{N}_4)$ is unstable for the reason that $\lambda_3 > 0$.
Proof:
(g) The secondary predator free equilibrium point at $E_{S7}(\overline{N}_1, \overline{N}_2, \overline{N}_3, 0)$ in matrix (5) then, the system of the variational matrix is

$$V(E_{S7}) = \begin{bmatrix} a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 & b_{12}\overline{N}_1 & -b_{13}\overline{N}_1 & -b_{14}\overline{N}_1 \\ b_{21}\overline{N}_2 & a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 & -b_{23}\overline{N}_2 & -b_{24}\overline{N}_2 \\ b_{31}\overline{N}_3 & b_{32}\overline{N}_3 & a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 & b_{34}\overline{N}_3 \\ 0 & 0 & 0 & a_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3 \end{bmatrix}$$

The characteristic equation of the above matrix is
$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0$ and $(a_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3) - \lambda_4 = 0$
The roots of the variational matrix is $\lambda_4 = (a_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3) > 0$

$$p_1 = (a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3) + (a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3) + (a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2)$$

$$p_2 = [(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3)(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2) + b_{23}\overline{N}_2b_{32}\overline{N}_3] + [(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3)(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2) + b_{13}\overline{N}_1b_{31}\overline{N}_3] + [(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3)(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3) - b_{12}\overline{N}_1b_{21}\overline{N}_2]$$

$$p_3 = (a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3)\begin{bmatrix} (a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3) \\ (a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2) + b_{23}\overline{N}_2b_{32}\overline{N}_3 \end{bmatrix} - b_{12}\overline{N}_1[b_{21}\overline{N}_2(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2) + b_{23}\overline{N}_2b_{31}\overline{N}_3] - b_{13}\overline{N}_1[b_{21}\overline{N}_2b_{32}\overline{N}_3 - b_{31}\overline{N}_3(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3)]$$

By using Routh-Hurwitz Criterion$p_1 > 0, p_2 > 0, p_3 > 0$, $p_1p_2 > 0$ and $(p_1p_2 - p_3) > 0$ then it is satisfied the system is asymptotically stable in $N_1N_2N_3$ plane. Otherwise it is unstable.
It's observed that $\lambda_4 > 0$ is always positive. Hence, the system of the state$E_{S7}(\overline{N}_1, \overline{N}_2, \overline{N}_3, 0)$ is unstable when λ_1, λ_2 and $\lambda_3$is positive $(\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0)$and saddle point exist when λ_1, λ_2 and $\lambda_3$is negative $(\lambda_1 < 0, \; \lambda_2 < 0, \lambda_3 < 0)$.
(h) The first prey free equilibrium point at $E_{S8}(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ in matrix (5) then, the system of the variational matrix is

$$V(E_{S8}) = \begin{bmatrix} a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4 & 0 & 0 & 0 \\ b_{21}\overline{N}_2 & a_2 - 2b_{22}\overline{N}_2 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4 & -b_{23}\overline{N}_2 & -b_{24}\overline{N}_2 \\ b_{31}\overline{N}_3 & b_{32}\overline{N}_3 & a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4 & b_{34}\overline{N}_3 \\ b_{41}\overline{N}_4 & b_{42}\overline{N}_4 & b_{43}\overline{N}_4 & a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3 \end{bmatrix}$$

The characteristic equation of the above matrix is
$(a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4) - \lambda_1 = 0$ and$\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0$
The roots of the variational matrix is $\lambda_1 = (a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4)$

$$q_1 = \left(a_2 - 2b_{22}\overline{N}_2 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) + \left(a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + (a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3)$$

$$q_2 = \left[\left(a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4\right]$$
$$+\left[\left(a_2 - 2b_{22}\overline{N}_2 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{24}\overline{N}_2 b_{42}\overline{N}_4\right]$$
$$+[\left(a_2 - 2b_{22}\overline{N}_2 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)\left(a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + b_{23}\overline{N}_2 b_{32}\overline{N}_3]$$

$$q_3 = \left(a_2 - 2b_{22}\overline{N}_2 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)\begin{bmatrix} \left(a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) \\ \left(a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4 \end{bmatrix}$$
$$+b_{23}\overline{N}_2\left[b_{32}\overline{N}_3\left(a_4 - 2b_{44}\overline{N}_4 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{42}\overline{N}_4\right]$$
$$-b_{24}[b_{32}\overline{N}_3 b_{43}\overline{N}_4 - b_{42}\overline{N}_4(a_3 - 2b_{33}\overline{N}_3 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4)]$$

By using Routh-Hurwitz Criterion $q_1 > 0, q_2 > 0, q_3 > 0, q_1 q_2 > 0$ and $(q_1 q_2 - q_3) > 0$ then it is satisfied the system is asymptotically stable in $N_2 N_3 N_4$ plane. Otherwise it is unstable.

It's observed that $\lambda_1$is positive (ie $\lambda_1 > 0$),if $\left(a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right) > 0$,. Hence, the system of the state E_{S8} $(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ is unstable whereasλ_2 ,λ_3 and $\lambda_4$are positive $(\lambda_2 > 0, \lambda_3 > 0, \ \lambda_4 > 0)$and saddle point exists whereasλ_2 ,λ_3 and $\lambda_4$are negative$(\lambda_2 < 0, \lambda_3 < 0, \lambda_4 < 0)$.

It's observed that λ_1 is negative (ie., $\lambda_1 < 0$),if$\left(a_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right) < 0$. Hence, the system of the state E_{S8} $(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ is asymptotically stable whereasλ_2 ,λ_3 and $\lambda_4$is also negative$(\lambda_2 < 0, \lambda_3 < 0, \ \lambda_4 < 0)$. Otherwise it is unstable and saddle point exist whereas λ_2 ,λ_3 and $\lambda_4$are positive$(\lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0)$.

(i) The second prey extinct equilibrium point at E_{S9} $(\overline{N}_1, 0, \overline{N}_3, \overline{N}_4)$ in matrix (5) then, the system of the variational matrix is

$$V(E_{S9}) = \begin{bmatrix} a_1 - 2b_{11}\overline{N}_1 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4 & b_{12}\overline{N}_1 & -b_{13}\overline{N}_1 & -b_{14}\overline{N}_1 \\ 0 & a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4 & 0 & 0 \\ b_{31}\overline{N}_3 & b_{32}\overline{N}_3 & a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4 & b_{34}\overline{N}_3 \\ b_{41}\overline{N}_4 & b_{42}\overline{N}_4 & b_{43}\overline{N}_4 & a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3 \end{bmatrix}$$

The characteristic equation of the above matrix is

$(a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4) - \lambda_2 = 0$ and $\lambda^3 + r_1\lambda^2 + r_2\lambda + r_3 = 0$

The roots of the variational matrix is $\lambda_2 = \left(a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)$

$$r_1 = (a_1 - 2b_{11}\overline{N}_1 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4) + \left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4\right) + (a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3)$$

$$r_2 = \left[\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4\right]$$
$$+\left[\left(a_1 - 2b_{11}\overline{N}_1 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3\right) + b_{14}\overline{N}_1 b_{41}\overline{N}_4\right]$$
$$+[\left(a_1 - 2b_{11}\overline{N}_1 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4\right) + b_{13}\overline{N}_1 b_{31}\overline{N}_3]$$

$$r_3 = \left(a_1 - 2b_{11}\overline{N}_1 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\begin{bmatrix} \left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4\right) \\ \left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4 \end{bmatrix}$$
$$+b_{13}\overline{N}_1\left[b_{31}\overline{N}_3\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{41}\overline{N}_4\right]$$
$$-b_{14}\overline{N}_1[b_{31}\overline{N}_3 b_{43}\overline{N}_4 - b_{41}\overline{N}_4(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{34}\overline{N}_4)]$$

By using Routh-Hurwitz Criterion$r_1 > 0, r_2 > 0, r_3 > 0, \ \ r_1 r_2 > 0$ and $(r_1 r_2 - r_3) > 0$ then it is satisfied then the system is asymptotically stable in $N_1 N_3 N_4$ plane. Otherwise it is unstable.

If $\left(a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) > 0$, it's observed that $\lambda_2$is positive (ie.,$\lambda_2 > 0$). Hence, the system of the state E_{S9} $(\overline{N}_1, 0, \overline{N}_3, \overline{N}_4)$is unstable whereas λ_1 ,λ_3 and $\lambda_4$are positive $(\lambda_1 > 0, \lambda_3 > 0, \ \lambda_4 > 0)$and saddle point exists whereasλ_1 ,λ_3 and $\lambda_4$are negative $(\lambda_1 < 0, \lambda_3 < 0, \lambda_4 < 0)$.

If $\left(a_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) < 0$,It's observed that λ_2 is negative (ie., $\lambda_2 < 0$), Hence, the system of the state E_{S9} $(\overline{N}_1, 0, \overline{N}_3, \overline{N}_4)$is asymptotically stable whereas λ_1 ,λ_3 and $\lambda_4$is also negative$(\lambda_1 < 0, \lambda_3 < 0, \lambda_4 < 0)$.

Otherwise, it is unstable and saddle point exist whereas λ_1 ,λ_3 and $\lambda_4$are positive $(\lambda_1 > 0, \lambda_3 > 0, \ \lambda_4 > 0)$.

(j) The primary predator extinct equilibrium point at E_{S10} $(\overline{N}_1, \overline{N}_2, 0, \overline{N}_4)$ in matrix (5) then, the system of the variational matrix is

$$V(E_{S10}) = \begin{bmatrix} a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{14}\overline{N}_4 & b_{12}\overline{N}_1 & -b_{13}\overline{N}_1 & -b_{14}\overline{N}_1 \\ b_{21}\overline{N}_2 & a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4 & -b_{23}\overline{N}_2 & -b_{24}\overline{N}_2 \\ 0 & 0 & a_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4 & 0 \\ b_{41}\overline{N}_4 & b_{42}\overline{N}_4 & b_{43}\overline{N}_4 & a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 \end{bmatrix}$$

The characteristic equation of the above matrix is

$(a_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4) - \lambda_3 = 0$ and $\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0$

$$\lambda_3 = \left(a_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) > 0$$

$$s_1 = (a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{14}\overline{N}_4) + \left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4\right) + (a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2)$$

$$\begin{aligned} s_2 = {} & \left[\left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2\right) + b_{24}\overline{N}_2 b_{42}\overline{N}_4\right] \\ & + \left[\left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{14}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2\right) + b_{14}\overline{N}_1 b_{41}\overline{N}_4\right] \\ & + \left[\left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{14}\overline{N}_4\right)\left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4\right) - b_{12}\overline{N}_1 b_{21}\overline{N}_2\right] \end{aligned}$$

$$\begin{aligned} s_3 = {} & \left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{14}\overline{N}_4\right)\begin{bmatrix} \left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4\right) \\ \left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2\right) + b_{24}\overline{N}_2 b_{42}\overline{N}_4 \end{bmatrix} \\ & - b_{12}\overline{N}_1\left[b_{21}\overline{N}_2\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2\right) + b_{24}\overline{N}_2 b_{41}\overline{N}_4\right] \\ & - b_{14}\overline{N}_1\left[b_{21}\overline{N}_2 b_{42}\overline{N}_4 - b_{41}\overline{N}_4(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{24}\overline{N}_4)\right] \end{aligned}$$

By using Routh Hurwitz Criterion $s_1 > 0, s_2 > 0, s_3 > 0$, $s_1 s_2 > 0$ and $(s_1 s_2 - s_3) > 0$ then it is satisfied then the system is asymptotically stable in $N_1N_2N_4$ plane. Otherwise it is unstable.

It's observed that $\lambda_3 > 0$ is always positive. Hence, the system of the state E_{S10} $(\overline{N}_1, \overline{N}_2, 0, \overline{N}_4)$ is unstable when λ_1, λ_2 and λ_4 are positive $(\lambda_1 > 0, \lambda_2 > 0, \lambda_4 > 0)$ and saddle point exist when λ_1, λ_2 and λ_4 are negative (ie., $\lambda_1 < 0, \lambda_2 < 0, \lambda_4 < 0$).

Theorem: 6.4

The coexistence steady state of the dynamic system (5) is asymptotically stable for the positive interior equilibrium point at $E_{P1}(\overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ if $h_1 > 0, h_1h_2 - h_3 > 0$, $(h_1h_2h_3 - h_3^2 - h_1^2h_4) > 0$, and $h_4 > 0$ holds.

Proof: All the species are exist at E_{P1} $(\overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ in matrix (5) then, the system of the variational matrix is

$$V(E_{P1}) = \begin{vmatrix} a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4 & b_{12}\overline{N}_1 & -b_{13}\overline{N}_1 & -b_{14}\overline{N}_1 \\ b_{21}\overline{N}_2 & a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4 & -b_{23}\overline{N}_2 & -b_{24}\overline{N}_2 \\ b_{31}\overline{N}_3 & b_{32}\overline{N}_3 & a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4 & b_{34}\overline{N}_3 \\ b_{41}\overline{N}_4 & b_{42}\overline{N}_4 & b_{43}\overline{N}_4 & a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3 \end{vmatrix}$$

The characteristic equation of the above matrix is $\lambda^4 + h_1\lambda^3 + h_2\lambda^2 + h_3\lambda + h_4 = 0$

$$\begin{aligned} h_1 = {} & \left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right) + \left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) \\ & + \left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + (a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3) \end{aligned}$$

$$\begin{aligned} h_2 = {} & \left[\left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) - b_{12}\overline{N}_1 b_{21}\overline{N}_2\right] \\ & + \left[\left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + b_{23}\overline{N}_2 b_{32}\overline{N}_3\right] \\ & + \left[\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4\right] \\ & + \left[\left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) + b_{14}\overline{N}_1 b_{41}\overline{N}_4\right] \\ & + \left[\left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + b_{13}\overline{N}_1 b_{31}\overline{N}_3\right] \\ & + \left[\left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right)\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) + b_{24}\overline{N}_2 b_{42}\overline{N}_4\right] \end{aligned}$$

$$\begin{aligned} h_3 = {} & \left(a_1 - 2b_{11}\overline{N}_1 + b_{12}\overline{N}_2 - b_{13}\overline{N}_3 - b_{14}\overline{N}_4\right)\begin{bmatrix} \left(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4\right) \\ \left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) \\ + b_{23}\overline{N}_2 b_{32}\overline{N}_3 \end{bmatrix} \\ & - b_{12}\overline{N}_1\left[b_{21}\overline{N}_2\left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) + b_{23}\overline{N}_2 b_{31}\overline{N}_3\right] \\ & - b_{13}\overline{N}_1\left[b_{21}\overline{N}_2 b_{32}\overline{N}_3 - b_{31}\overline{N}_3(a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4)\right] \\ & + (a_2 - 2b_{22}\overline{N}_2 + b_{21}\overline{N}_1 - b_{23}\overline{N}_3 - b_{24}\overline{N}_4)\begin{bmatrix} \left(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4\right) \\ \left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right) - b_{34}\overline{N}_3 b_{43}\overline{N}_4 \end{bmatrix} \\ & + b_{23}\overline{N}_2\left[\left(b_{32}\overline{N}_3\left(a_4 - 2b_{44}\overline{N}_4 + b_{41}\overline{N}_1 + b_{42}\overline{N}_2 + b_{43}\overline{N}_3\right)\right) - b_{34}\overline{N}_3 b_{42}\overline{N}_4\right] \\ & - b_{24}\overline{N}_2\left[(b_{32}\overline{N}_3 b_{43}\overline{N}_4 - b_{42}\overline{N}_4(a_3 - 2b_{33}\overline{N}_3 + b_{31}\overline{N}_1 + b_{32}\overline{N}_2 + b_{34}\overline{N}_4))\right] \end{aligned}$$

$$+\left(a_1-2b_{11}\overline{N}_1+b_{12}\overline{N}_2-b_{13}\overline{N}_3-b_{14}\overline{N}_4\right)\begin{bmatrix}(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)\\(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)\\-b_{34}\overline{N}_3b_{43}\overline{N}_4\end{bmatrix}$$
$$+b_{13}\overline{N}_1[b_{31}\overline{N}_3(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{41}\overline{N}_4]$$
$$-b_{14}\overline{N}_1[b_{31}\overline{N}_3b_{43}\overline{N}_4-b_{41}\overline{N}_4(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)$$
$$+(a_1-2b_{11}\overline{N}_1+b_{12}\overline{N}_2-b_{13}\overline{N}_3-b_{14}\overline{N}_4)\begin{bmatrix}(a_2-2b_{22}\overline{N}_2+b_{21}\overline{N}_1-b_{23}\overline{N}_3-b_{24}\overline{N}_4)\\(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)\\+b_{24}\overline{N}_2b_{42}\overline{N}_4\end{bmatrix}$$
$$-b_{12}\overline{N}_1\left[\left(b_{21}\overline{N}_2(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)\right)+b_{24}\overline{N}_2b_{41}\overline{N}_4\right]$$
$$-b_{14}\overline{N}_1[(b_{21}\overline{N}_2b_{42}\overline{N}_4-b_{41}\overline{N}_4(a_2-2b_{22}\overline{N}_2+b_{21}\overline{N}_1-b_{23}\overline{N}_3-b_{24}\overline{N}_4))]$$

$$h_4=\left(a_1-2b_{11}\overline{N}_1+b_{12}\overline{N}_2-b_{13}\overline{N}_3-b_{14}\overline{N}_4\right)$$
$$\begin{bmatrix}(a_2-2b_{22}\overline{N}_2+b_{21}\overline{N}_1-b_{23}\overline{N}_3-b_{24}\overline{N}_4)\begin{bmatrix}(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)\\(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{43}\overline{N}_4\end{bmatrix}\\+b_{23}\overline{N}_2[b_{32}\overline{N}_3(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{42}\overline{N}_4]\\-b_{24}\overline{N}_2[b_{32}\overline{N}_3b_{43}\overline{N}_4-b_{42}\overline{N}_4(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)]\end{bmatrix}$$
$$-b_{12}\overline{N}_1\begin{bmatrix}b_{21}\overline{N}_2[(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)\\(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{43}\overline{N}_4]\\+b_{23}\overline{N}_2\left[\left(b_{31}\overline{N}_3(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)\right)-b_{34}\overline{N}_3b_{41}\overline{N}_4\right]\\-b_{24}\overline{N}_2\left[\left(b_{31}\overline{N}_3b_{43}\overline{N}_4-b_{41}\overline{N}_4(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)\right)\right]\end{bmatrix}$$
$$-b_{13}\overline{N}_1\begin{bmatrix}[b_{21}\overline{N}_2(b_{32}\overline{N}_3(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{42}\overline{N}_4)]\\-(a_2-2b_{22}\overline{N}_2+b_{21}\overline{N}_1-b_{23}\overline{N}_3-b_{24}\overline{N}_4)\\[b_{31}\overline{N}_3(a_4-2b_{44}\overline{N}_4+b_{41}\overline{N}_1+b_{42}\overline{N}_2+b_{43}\overline{N}_3)-b_{34}\overline{N}_3b_{41}\overline{N}_4]\\-b_{24}\overline{N}_2[(b_{31}\overline{N}_3b_{42}\overline{N}_4-b_{32}\overline{N}_3b_{41}\overline{N}_4)]\end{bmatrix}$$
$$+b_{14}\overline{N}_1\begin{bmatrix}\left[b_{21}\overline{N}_2\left(b_{32}\overline{N}_3b_{43}\overline{N}_4-b_{42}\overline{N}_4(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)\right)\right]\\-(a_2-2b_{22}\overline{N}_2+b_{21}\overline{N}_1-b_{23}\overline{N}_3-b_{24}\overline{N}_4)\\[b_{31}\overline{N}_3b_{43}\overline{N}_4-b_{41}\overline{N}_4(a_3-2b_{33}\overline{N}_3+b_{31}\overline{N}_1+b_{32}\overline{N}_2+b_{34}\overline{N}_4)]\\-b_{23}\overline{N}_2[(b_{31}\overline{N}_3b_{42}\overline{N}_4-b_{32}\overline{N}_3b_{41}\overline{N}_4)]\end{bmatrix}$$

According to Routh Hurwitz Criterion is locally asymptotically stable at the equilibrium point $E_{P1}(\overline{N}_1,\overline{N}_2,\overline{N}_3,\overline{N}_4)$. If $h_1>0, h_1h_2-h_3>0$, $(h_1h_2h_3-h_3^2-h_1^2h_4)>0$ and $h_4>0$ If any one of the four conditions is violated, then the systems will no longer stable at the equilibrium point E_{P1}. Any one of our considered two predators has a great effect on the two prey populations and on the rest considered predator species. So, a small change in their populations can make a great impact on the rest of the populations.

7. Global Stability Analysis

It states that stability analysis is performed using the Lyapunov function to determine the global stability relative to the nearest equilibrium point of the state.

Theorem: 7.1

The global steady state of the dynamic system (1) shows the stability condition for the following Semi interior equilibrium point when two of the species extinct state.

(i) The semi interior equilibrium state $E_{S1}(\overline{N}_1,\overline{N}_2,0{,}0)$ of the four species is globally asymptotically stable, if $\left(b_{11}>\frac{b_{12}+b_{21}}{2}\right)$ *and* $\left(b_{22}>\frac{b_{12}+b_{21}}{2}\right)$.

(ii) If $\left(b_{22}+\frac{b_{23}-b_{32}}{2}\right)>0$ *and* $\left(b_{33}+\frac{b_{23}-b_{32}}{2}\right)>0$, then the semi interior equilibrium state $E_{S2}(0,\overline{N}_2,\overline{N}_3,0)$ of

the four species is globally asymptotically stable.
(iii) If the condition $\left(b_{33} > \frac{b_{34}+b_{43}}{2}\right)$ *and* $\left(b_{44} > \frac{b_{34}+b_{43}}{2}\right)$ hold, then the semi interior equilibrium state $E_{S3}(0,0,\overline{N}_3,\overline{N}_4)$of the four species exhibits globally asymptotically stable.
(iv) If the condition $\left(b_{11} + \frac{b_{14}-b_{41}}{2}\right) > 0$ *and* $\left(b_{44} + \frac{b_{14}-b_{41}}{2}\right) > 0$ is satisfied, then the semi interior equilibrium
state $E_{S4}(\overline{N}_1,0,0,\overline{N}_4)$ of the four species is globally asymptotically stable.
(v) If $\left(b_{11} + \frac{b_{13}-b_{31}}{2}\right) > 0$ *and* $\left(b_{33} + \frac{b_{13}-b_{31}}{2}\right) > 0$, then the semi interior equilibrium state $E_{S5}(\overline{N}_1,0,\overline{N}_3,0)$ of
the four species is globally asymptotically stable.
(vi) If both $\left(b_{22} + \frac{b_{24}-b_{42}}{2}\right) > 0$ *and* $\left(b_{44} + \frac{b_{24}-b_{42}}{2}\right) > 0$, then the semi interior equilibrium state $E_{S6}(0,\overline{N}_2,0,\overline{N}_4)$ of the four species is established as globally asymptotically stable.

(i) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_2) = (N_1 - \overline{N}_1) - \overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right] + (N_2 - \overline{N}_2) - \overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right]$$

$$\frac{dV}{dt} = \frac{dN_1}{dt}\left[1 - \frac{\overline{N}_1}{N_1}\right] + \frac{dN_2}{dt}\left[1 - \frac{\overline{N}_2}{N_2}\right]$$

$$= [N_1 - \overline{N}_1](a_1 - b_{11}N_1 + b_{12}N_2) + [N_2 - \overline{N}_2](a_2 - b_{22}N_2 + b_{21}N_1)$$

Substitute $a_1 = b_{11}\overline{N}_1 - b_{12}\overline{N}_2$, $a_2 = b_{22}\overline{N}_2 - b_{21}\overline{N}_1$

$$= -[N_1 - \overline{N}_1]^2\left(b_{11} - \frac{b_{12}+b_{21}}{2}\right) - [N_2 - \overline{N}_2]^2\left(b_{22} - \frac{b_{12}+b_{21}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad if\left(b_{11} > \frac{b_{12}+b_{21}}{2}\right) and \left(b_{22} > \frac{b_{12}+b_{21}}{2}\right)$$

Thus, the given system $E_{S1}(\overline{N}_1,\overline{N}_2,0,0)$is globally asymptotically stable.

(ii) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_2,\overline{N}_3) = (N_2 - \overline{N}_2) - \overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right] + (N_3 - \overline{N}_3) - \overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]$$

$$\frac{dV}{dt} = \frac{dN_2}{dt}\left[1 - \frac{\overline{N}_2}{N_2}\right] + \frac{dN_3}{dt}\left[1 - \frac{\overline{N}_3}{N_3}\right]$$

$$= [N_2 - \overline{N}_2](a_2N_2 - b_{22}N_2 - b_{23}N_3) + [N_3 - \overline{N}_3](a_3 - b_{33}N_3 + b_{32}N_2)$$

Substitute $a_2 = b_{22}\overline{N}_2 + b_{23}\overline{N}_3$, $a_3 = b_{33}\overline{N}_3 - b_{32}\overline{N}_2$

$$= -[N_2 - \overline{N}_2]^2\left(b_{22} + \frac{b_{23}-b_{32}}{2}\right) - [N_3 - \overline{N}_3]^2\left(b_{33} + \frac{b_{23}-b_{32}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad \left(b_{22} + \frac{b_{23}-b_{32}}{2}\right) > 0 \; and \; \left(b_{33} + \frac{b_{23}-b_{32}}{2}\right) > 0$$

Hence the system described by $E_{S2}(0,\overline{N}_2,\overline{N}_3,0)$attains globally asymptotically stable.

(iii) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_3,\overline{N}_4) = (N_3 - \overline{N}_3) - \overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right] + (N_4 - \overline{N}_4) - \overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt} = \frac{dN_3}{dt}\left[1 - \frac{\overline{N}_3}{N_3}\right] + \frac{dN_4}{dt}\left[1 - \frac{\overline{N}_4}{N_4}\right]$$

$$= [N_3 - \overline{N}_3](a_3 - b_{33}N_3 + b_{34}N_4) + [N_4 - \overline{N}_4](a_4 - b_{44}N_4 + b_{43}N_3)$$

Substitute $a_3 = b_{33}\overline{N}_3 - b_{34}\overline{N}_4$, $a_4 = b_{44}\overline{N}_4 - b_{43}\overline{N}_3$

$$= -[N_3 - \overline{N}_3]^2\left(b_{33} - \frac{b_{34}+b_{43}}{2}\right) - [N_4 - \overline{N}_4]^2\left(b_{44} - \frac{b_{34}+b_{43}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad if\left(b_{33} > \frac{b_{34}+b_{43}}{2}\right) and \left(b_{44} > \frac{b_{34}+b_{43}}{2}\right)$$

Therefore, the system characterized by $E_{S3}(0,0,\overline{N}_3,\overline{N}_4)$achievesglobally asymptotically stable.

(iv) Proof:

Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_4) = (N_1 - \overline{N}_1) - \overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right] + (N_4 - \overline{N}_4) - \overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt} = \frac{dN_1}{dt}\left[1 - \frac{\overline{N}_1}{N_1}\right] + \frac{dN_4}{dt}\left[1 - \frac{\overline{N}_4}{N_4}\right]$$

$$= [N_1 - \overline{N}_1](a_1 - b_{11}N_1 - b_{14}N_4) + [N_4 - \overline{N}_4](a_4 - b_{44}N_4 + b_{41}N_1)$$

Substitute $a_1 = b_{11}\overline{N}_1 + b_{14}\overline{N}_4$, $a_4 = b_{44}\overline{N}_4 - b_{41}\overline{N}_1$

$$= -[N_1 - \overline{N}_1]^2\left(b_{11} + \frac{b_{14}-b_{41}}{2}\right) - [N_4 - \overline{N}_4]^2\left(b_{44} + \frac{b_{14}-b_{41}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad if\left(b_{11} + \frac{b_{14}-b_{41}}{2}\right) > 0 \; and \; \left(b_{44} + \frac{b_{14}-b_{41}}{2}\right) > 0$$

Hence, the system represented by $E_{S4}(\overline{N}_1,0,0,\overline{N}_4)$attains globally asymptotically stable.

(v) Proof:

Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_3) = (N_1 - \overline{N}_1) - \overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right] + (N_3 - \overline{N}_3) - \overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]$$

$$\frac{dV}{dt} = \frac{dN_1}{dt}\left[1 - \frac{\overline{N}_1}{N_1}\right] + \frac{dN_3}{dt}\left[1 - \frac{\overline{N}_3}{N_3}\right]$$

$$= [N_1 - \overline{N}_1](a_1 - b_{11}N_1 - b_{13}N_3) + [N_3 - \overline{N}_3](a_3 - b_{33}N_3 + b_{31}N_1)$$

Substitute $a_1 = b_{11}\overline{N}_1 + b_{13}\overline{N}_3$, $\quad a_3 = b_{33}\overline{N}_3 - b_{31}\overline{N}_1$

$$= -[N_1 - \overline{N}_1]^2\left(b_{11} + \frac{b_{13}-b_{31}}{2}\right) - [N_3 - \overline{N}_3]^2\left(b_{33} + \frac{b_{13}-b_{31}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad if\;\left(b_{11} + \frac{b_{13}-b_{31}}{2}\right) > 0 \; and \; \left(b_{33} + \frac{b_{13}-b_{31}}{2}\right) > 0$$

Thus, the system described by $E_{S5}(\overline{N}_1, 0, \overline{N}_3, 0)$ attains globally asymptotically stable.

(vi) Proof:

Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_2,\overline{N}_4) = (N_2 - \overline{N}_2) - \overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right] + (N_4 - \overline{N}_4) - \overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt} = \frac{dN_2}{dt}\left[1 - \frac{\overline{N}_2}{N_2}\right] + \frac{dN_4}{dt}\left[1 - \frac{\overline{N}_4}{N_4}\right]$$

$$= [N_2 - \overline{N}_2](a_2 - b_{22}N_2 - b_{24}N_4) + [N_4 - \overline{N}_4](a_4 - b_{44}N_4 + b_{42}N_2)$$

Substitute $a_2 = b_{22}\overline{N}_2 + b_{24}\overline{N}_4$, $\quad a_4 = b_{44}\overline{N}_4 - b_{42}\overline{N}_2$

$$= -[N_2 - \overline{N}_2]^2\left(b_{22} + \frac{b_{24}-b_{42}}{2}\right) - [N_4 - \overline{N}_4]^2\left(b_{44} + \frac{b_{24}-b_{42}}{2}\right)$$

$$\frac{dV}{dt} < 0\,, \quad if\left(b_{22} + \frac{b_{24}-b_{42}}{2}\right) > 0 \; and \left(b_{44} + \frac{b_{24}-b_{42}}{2}\right) > 0$$

Thus, the system denoted by $E_{S6}(0, \overline{N}_2, 0, \overline{N}_4)$demonstrates globally asymptotically stable.

Theorem: 7.2

The global steady state of dynamic system (1) shows the stability condition when one of the species reaches extinction at the specified semi-interior equilibrium point.

(i) The semi interior equilibrium state $E_{S7}(\overline{N}_1, \overline{N}_2, \overline{N}_3, 0)$of the four species is globally asymptotically Stable, $if\left(b_{11} + \frac{b_{13}-b_{12}-b_{21}-b_{31}}{2}\right) > 0,\ \left(b_{22} + \frac{b_{13}-b_{12}-b_{21}-b_{31}}{2}\right) > 0\; and \left(b_{33} + \frac{b_{31}-b_{32}-b_{13}-b_{23}}{2}\right) > 0.$

(ii) If the condition $\left(b_{22} + \frac{b_{23}+b_{24}-b_{32}-b_{42}}{2}\right) > 0,\ \left(b_{33} + \frac{b_{23}-b_{32}-b_{34}-b_{43}}{2}\right) > 0\; and \left(b_{44} + \frac{b_{24}-b_{42}-b_{43}-b_{34}}{2}\right) > 0$

is satisfied, then the semi interior equilibrium state $E_{S8}(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$of the four species is globally

asymptotically stable.
(iii) The semi interior equilibrium state $E_{S9}(\overline{N}_1, 0, \overline{N}_3, \overline{N}_4)$of the four species is globally asymptotically stable, $if\left(b_{11}+\frac{b_{13}+b_{14}-b_{31}-b_{41}}{2}\right)>0,\ \left(b_{33}+\frac{b_{13}-b_{31}-b_{34}-b_{43}}{2}\right)>0\ and\left(b_{44}+\frac{b_{14}-b_{41}-b_{43}-b_{34}}{2}\right)>0.$
(iv) If the condition $\left(b_{11}+\frac{b_{14}-b_{12}-b_{21}-b_{41}}{2}\right)>0,\ \left(b_{22}+\frac{b_{24}-b_{21}-b_{12}-b_{42}}{2}\right)>0\ and\left(b_{44}+\frac{b_{14}+b_{24}-b_{41}-b_{42}}{2}\right)>0$
holds, then the semi interior equilibrium state $E_{S10}(\overline{N}_1, \overline{N}_2, 0, \overline{N}_4)$of the four species is globally asymptotically stable.

(i) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_3)=(N_1-\overline{N}_1)-\overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right]+(N_2-\overline{N}_2)-\overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right]+(N_3-\overline{N}_3)-\overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]$$

$$\frac{dV}{dt}=\frac{dN_1}{dt}\left[1-\frac{\overline{N}_1}{N_1}\right]+\frac{dN_2}{dt}\left[1-\frac{\overline{N}_2}{N_2}\right]+\frac{dN_3}{dt}\left[1-\frac{\overline{N}_3}{N_3}\right]$$

$$=[N_1-\overline{N}_1](a_1-b_{11}N_1+b_{12}N_2-b_{13}N_3)+[N_2-\overline{N}_2](a_2-b_{22}N_2+b_{21}N_1b_{23}N_3)$$
$$+[N_3-\overline{N}_3](a_3-b_{33}N_3+b_{31}N_1+b_{32}N_2)$$

Substitute $a_1=b_{11}\overline{N}_1-b_{12}\overline{N}_2+b_{13}\overline{N}_3$, $a_2=b_{22}\overline{N}_2-b_{21}\overline{N}_1+b_{23}\overline{N}_3$, $a_3=b_{33}\overline{N}_3-b_{31}\overline{N}_1-b_{32}\overline{N}_2$

$$=-[N_1-\overline{N}_1]^2\left(b_{11}+\frac{b_{13}-b_{12}-b_{21}-b_{31}}{2}\right)-[N_2-\overline{N}_2]^2\left(b_{22}+\frac{b_{23}-b_{12}-b_{21}-b_{32}}{2}\right)$$
$$-[N_3-\overline{N}_3]^2\left(b_{33}+\frac{b_{13}+b_{23}-b_{31}-b_{32}}{2}\right)$$

$$\frac{dV}{dt}<0\ ,\quad if\left(b_{11}+\frac{b_{13}-b_{12}-b_{21}-b_{31}}{2}\right)>0,$$
$$\left(b_{22}+\frac{b_{13}-b_{12}-b_{21}-b_{31}}{2}\right)>0\ and\left(b_{33}+\frac{b_{31}-b_{32}-b_{13}-b_{23}}{2}\right)>0$$

Hence, the system denoted by $(\overline{N}_1, \overline{N}_2, \overline{N}_3, 0)$exhibit globally asymptotically stable.

(ii) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_2,\overline{N}_3,\overline{N}_4)=(N_2-\overline{N}_2)-\overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right]+(N_3-\overline{N}_3)-\overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]+(N_4-\overline{N}_4)-\overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt}=\frac{dN_2}{dt}\left[1-\frac{\overline{N}_2}{N_2}\right]+\frac{dN_3}{dt}\left[1-\frac{\overline{N}_3}{N_3}\right]+\frac{dN_4}{dt}\left[1-\frac{\overline{N}_4}{N_4}\right]$$

$$=[N_2-\overline{N}_2](a_2-b_{22}N_2-b_{23}N_3-b_{24}N_4)+[N_3-\overline{N}_3](a_3-b_{33}N_3+b_{32}N_2+b_{34}N_4)$$
$$+\quad[N_4-\overline{N}_4](a_4-b_{44}N_4+b_{42}N_2+b_{43}N_3)$$

Substitute $a_2=b_{22}\overline{N}_2+b_{23}\overline{N}_3+b_{24}\overline{N}_4$, $a_3=b_{33}\overline{N}_3-b_{32}\overline{N}_2-b_{34}\overline{N}_4$, $a_4=b_{44}\overline{N}_4-b_{42}\overline{N}_2-b_{43}\overline{N}_3$

$$=\begin{bmatrix}-[N_2-\overline{N}_2]^2\left(b_{22}+\frac{b_{23}+b_{24}-b_{32}-b_{42}}{2}\right)-[N_3-\overline{N}_3]^2\left(b_{33}+\frac{b_{23}-b_{32}-b_{34}-b_{43}}{2}\right)\\ -[N_4-\overline{N}_4]^2\left(b_{44}+\frac{b_{24}-b_{42}-b_{43}-b_{34}}{2}\right)\end{bmatrix}$$

$$\frac{dV}{dt}<0\ ,\qquad if\left(b_{22}+\frac{b_{23}+b_{24}-b_{32}-b_{42}}{2}\right)>0,$$
$$\left(b_{33}+\frac{b_{23}-b_{32}-b_{34}-b_{43}}{2}\right)>0\ and\left(b_{44}+\frac{b_{24}-b_{42}-b_{43}-b_{34}}{2}\right)>o$$

As a result, the system characterized by$E_{S8}(0, \overline{N}_2, \overline{N}_3, \overline{N}_4)$ attains globally asymptotically stable.

(iii) Proof:
Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_3,\overline{N}_4)=(N_1-\overline{N}_1)-\overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right]+(N_3-\overline{N}_3)-\overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]+(N_4-\overline{N}_4)-\overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt}=\frac{dN_1}{dt}\left[1-\frac{\overline{N}_1}{N_1}\right]+\frac{dN_3}{dt}\left[1-\frac{\overline{N}_3}{N_3}\right]+\frac{dN_4}{dt}\left[1-\frac{\overline{N}_4}{N_4}\right]$$

$$=\left[N_1-\overline{N}_1\right](a_1-b_{11}N_1-b_{13}N_3-b_{14}N_4)+\left[N_3-\overline{N}_3\right](a_3-b_{33}N_3+b_{31}N_1+b_{34}N_4)+\left[N_4-\overline{N}_4\right](a_4-b_{44}N_4+b_{41}N_1+b_{43}N_3)$$

Substitute $a_1=b_{11}\overline{N}_1+b_{13}\overline{N}_3+b_{14}\overline{N}_4,\ a_3=b_{33}\overline{N_3}-b_{31}\overline{N}_1-b_{34}\overline{N}_4,\ a_4=b_{44}\overline{N}_4-b_{41}\overline{N}_1-b_{43}\overline{N}_3$

$$=\left[-\left[N_1-\overline{N}_1\right]^2\left(b_{11}+\frac{b_{13}+b_{14}-b_{31}-b_{41}}{2}\right)-\left[N_3-\overline{N}_3\right]^2\left(b_{33}+\frac{b_{13}-b_{31}-b_{34}-b_{43}}{2}\right)-\left[N_4-\overline{N}_4\right]^2\left(b_{44}+\frac{b_{14}-b_{41}-b_{43}-b_{34}}{2}\right)\right]$$

$$\frac{dV}{dt}<0,\quad if\left(b_{11}+\frac{b_{13}+b_{14}-b_{31}-b_{41}}{2}\right)>0,$$
$$\left(b_{33}+\frac{b_{13}-b_{31}-b_{34}-b_{43}}{2}\right)>0\ and\left(b_{44}+\frac{b_{14}-b_{41}-b_{43}-b_{34}}{2}\right)>0$$

Thus, the system describe by $E_{S9}(\overline{N}_1,0,\overline{N}_3,\overline{N}_4)$is globally asymptotically stable.

(iv) Proof:

Let us take the system of the semi interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_4)=(N_1-\overline{N}_1)-\overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right]+(N_2-\overline{N}_2)-\overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right]+(N_4-\overline{N}_4)-\overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]$$

$$\frac{dV}{dt}=\frac{dN_1}{dt}\left[1-\frac{\overline{N}_1}{N_1}\right]+\frac{dN_2}{dt}\left[1-\frac{\overline{N}_2}{N_2}\right]+\frac{dN_4}{dt}\left[1-\frac{\overline{N}_4}{N_4}\right]$$

$$=\left[N_1-\overline{N}_1\right](a_1-b_{11}N_1+b_{12}N_2-b_{14}N_4)+\left[N_2-\overline{N}_2\right](a_2-b_{22}N_2+b_{21}N_1-b_{24}N_4)+\left[N_4-\overline{N}_4\right](a_4-b_{44}N_4+b_{41}N_1+b_{42}N_2)$$

Substitute $a_1=b_{11}\overline{N}_1-b_{12}\overline{N}_2+b_{14}\overline{N}_4,\ a_2=b_{22}\overline{N}_2-b_{21}\overline{N}_1+b_{24}\overline{N}_4,\ a_4=b_{44}\overline{N}_4-b_{41}\overline{N}_1-b_{42}\overline{N}_2$

$$=\left[-\left[N_1-\overline{N}_1\right]^2\left(b_{11}+\frac{b_{14}-b_{12}-b_{21}-b_{41}}{2}\right)-\left[N_2-\overline{N}_2\right]^2\left(b_{22}+\frac{b_{24}-b_{21}-b_{12}-b_{42}}{2}\right)-\left[N_4-\overline{N}_4\right]^2\left(b_{44}+\frac{b_{14}+b_{24}-b_{41}-b_{42}}{2}\right)\right]$$

$$\frac{dV}{dt}<0,\quad if\left(b_{11}+\frac{b_{14}-b_{12}-b_{21}-b_{41}}{2}\right)>0,$$
$$\left(b_{22}+\frac{b_{24}-b_{21}-b_{12}-b_{42}}{2}\right)>0\ and\left(b_{44}+\frac{b_{14}+b_{24}-b_{41}-b_{42}}{2}\right)>0$$

Thus, the system, characterized by $E_{S10}(\overline{N}_1,\overline{N}_2,0,\overline{N}_4)$ attains globally asymptotically stable.

Theorem: 7.3

If the condition

$$\left[\left(b_{11}+\frac{b_{13}+b_{14}-b_{12}-b_{21}-b_{31}-b_{41}}{2}\right)>0,\left(b_{22}+\frac{b_{23}+b_{24}-b_{21}-b_{12}-b_{32}-b_{42}}{2}\right)>0,\left(b_{33}+\frac{b_{13}+b_{23}-b_{31}-b_{32}-b_{34}-b_{43}}{2}\right)>0\ \left(b_{44}+\frac{b_{14}+b_{24}-b_{41}-b_{42}-b_{43}-b_{34}}{2}\right)>0\right]$$

satisfied, then the positive interior equilibrium state $E_{P1}(\overline{N}_1,\overline{N}_2,\overline{N}_3,\overline{N}_4)$of the four species is globally asymptotically stable.

Proof:

Let us take the system of the positive interior equilibrium state by applying Lyapunov function

$$V(\overline{N}_1,\overline{N}_2,\overline{N}_3,\overline{N}_4)=\left[(N_1-\overline{N}_1)-\overline{N}_1 ln\left[\frac{N_1}{\overline{N}_1}\right]+(N_2-\overline{N}_2)-\overline{N}_2 ln\left[\frac{N_2}{\overline{N}_2}\right]+(N_3-\overline{N}_3)-\overline{N}_3 ln\left[\frac{N_3}{\overline{N}_3}\right]+(N_4-\overline{N}_4)-\overline{N}_4 ln\left[\frac{N_4}{\overline{N}_4}\right]\right]$$

$$\frac{dV}{dt}=\frac{dN_1}{dt}\left[1-\frac{\overline{N}_1}{N_1}\right]+\frac{dN_2}{dt}\left[1-\frac{\overline{N}_2}{N_2}\right]+\frac{dN_3}{dt}\left[1-\frac{\overline{N}_3}{N_3}\right]+\frac{dN_4}{dt}\left[1-\frac{\overline{N}_4}{N_4}\right]$$

Determine $\frac{dV}{dt}$ which follows

$$= \begin{bmatrix} [N_1 - \overline{N}_1](a_1 - b_{11}N_1 + b_{12}N_2 - b_{13}N_3 - b_{14}N_4) \\ +[N_2 - \overline{N}_2](a_2 - b_{22}N_2 + b_{21}N_1 - b_{23}N_3 - b_{24}N_4) \\ +[N_3 - \overline{N}_3](a_3 - b_{33}N_3 + b_{31}N_1 + b_{32}N_2 + b_{34}N_4) \\ +[N_4 - \overline{N}_4](a_4 - b_{44}N_4 + b_{41}N_1 + b_{42}N_2 + b_{43}N_3) \end{bmatrix}$$

Substitute $a_1 = b_{11}\overline{N}_1 - b_{12}\overline{N}_2 + b_{13}\overline{N}_3 + b_{14}\overline{N}_4$, $a_2 = b_{22}\overline{N}_2 - b_{21}\overline{N}_1 + b_{23}\overline{N}_3 + b_{24}\overline{N}_4$, $a_3 = b_{33}\overline{N}_3 - b_{31}\overline{N}_1 - b_{32}\overline{N}_2 - b_{34}\overline{N}_4$, $a_4 = b_{44}\overline{N}_4 - b_{41}\overline{N}_1 - b_{42}\overline{N}_2 - b_{43}\overline{N}_3$

$$= \begin{bmatrix} -[N_1 - \overline{N}_1]^2 \left(b_{11} + \frac{b_{13} + b_{14} - b_{12} - b_{21} - b_{31} - b_{41}}{2}\right) \\ -[N_2 - \overline{N}_2]^2 \left(b_{22} + \frac{b_{23} + b_{24} - b_{21} - b_{12} - b_{32} - b_{42}}{2}\right) \\ -[N_3 - \overline{N}_3]^2 \left(b_{33} + \frac{b_{13} + b_{23} - b_{31} - b_{32} - b_{34} - b_{43}}{2}\right) \\ -[N_4 - \overline{N}_4]^2 \left(b_{44} + \frac{b_{14} + b_{24} - b_{41} - b_{42} - b_{43} - b_{34}}{2}\right) \end{bmatrix}$$

$\frac{dV}{dt} < 0$,

$$\begin{bmatrix} \text{if } \left(b_{11} + \frac{b_{13} + b_{14} - b_{12} - b_{21} - b_{31} - b_{41}}{2}\right) > 0, \left(b_{22} + \frac{b_{23} + b_{24} - b_{21} - b_{12} - b_{32} - b_{42}}{2}\right) > 0, \\ \left(b_{33} + \frac{b_{13} + b_{23} - b_{31} - b_{32} - b_{34} - b_{43}}{2}\right) > 0, \left(b_{44} + \frac{b_{14} + b_{24} - b_{41} - b_{42} - b_{43} - b_{34}}{2}\right) > 0 \end{bmatrix}$$

Therefore, the system, as described by $E_{P1}(\overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{N}_4)$achieves globally asymptotically stable.

8. Conclusion

This research explores a four-species biological model, emphasizing mutualism and predation interactions. The analysis aims to provide a comprehensive understanding of stability properties at various equilibrium points in the dynamic system. The variational matrix analysis, conducted from E_T toE_{P1}, reveals stability conditions dependent on the relationships between coefficients. The system is deemed asymptotically stable if specific conditions are met; otherwise, it is considered unstable. The study examines the coexistence of species, shedding light on their survival dynamics within the environment. Prey exhibit extended survival when they have access to plentiful food resources that are shared among the individuals within the species group in the region. Similarly, predators can sustain themselves to some extent by fulfilling their dietary needs through mutualistic interactions with either the same or other types of predatory species when prey is unavailable. However, their survival is significantly compromised in the complete absence of prey. The study also notes a potential reduction in species population density when prey and predators directly interact.
Validation involved a comprehensive analysis of species coexistence stability at each equilibrium phase, both locally and globally, using the Routh-Hurwitz criterion and Lyapunov function. This approach deepens our understanding of the delicate balance these species achieve in their ecological niche, providing insights into coexistence and competition dynamics within diverse ecosystems. Ultimately, the research significantly contributes to the broader field of ecological studies, illuminating the intricate complexities inherent in interactions within diverse ecosystems

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