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A Note On Strongly Gorenstein $\mathcal{X}$-Flat Modules

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Abstract

Mao and Ding introduced the concept of injective modules. D. Bennis and N. Mahdou introduced and studied the concept of strongly Gorenstein projective and injective modules. In this article, we have introduced and examined strongly Gorenstein-flat modules, which are the generalizations of strongly flat modules. Further, we have linked them with the strongly Gorenstein-projective modules.

1. INTRODUCTION

In this article, R denotes an associative ring with identity and all R-modules, if not specified otherwise, are left R-modules. R-Mod denotes the category of left R-modules.
Let $\mathcal{X}$ be a class of left R-modules. Mao and Ding in [Mao] have introduced the concept of $\mathcal{X}$ -injective modules. A left R-module M is called$\mathcal{X}$-injective if $Ext_R^1(X, M) = 0$ for all left R-modules $X \in \mathcal{X}$. We have introduced the concept of $\mathcal{X}$ –projective modules. A left R-module M is called $\mathcal{X}$-projective if $Ext_R^1(M, X) = 0$ for all left R-modules $X \in \mathcal{X}$.
D. Bennis and N. Mahdou have introduced and studied the concept of Strongly Gorenstein projective and injective modules. We have examined and studied strongly Gorenstein $\boldsymbol{\mathcal{X}}$ -flat modules, which is the generalization of strongly flat modules. Further we have linked them with the strongly Gorenstein $\boldsymbol{\mathcal{X}}$-projective modules.

2. PRELIMINARIES

In this section, we have recalled some of the known definitions and terminology that will be used in the rest of the work.
Given a class $\mathbb{C}$ of left R -modules, is written as

$$\mathbb{C}^{\perp} = \{N \in R - Mod/Ext_R^1(M, N) = 0, \forall M \in \mathbb{C}\}$$
$${}^{\perp}\mathbb{C} = \{N \in R - Mod/Ext_R^1(N, M) = 0, \forall M \in \mathbb{C}\}$$

For an R-module M, $fd(M)$ denote the flat dimension of M and $id(M\ )$ denote the injective

dimension of M. The $\mathcal{X}^{\perp}$ coresolution dimension of M, denoted by $cores.dim_{\mathcal{X}^{\perp}}(M)$, is to be the smallest nonnegative integer n such that $Ext_R^{n+1}(A, M) = 0$ for all R-modules $A \in \mathcal{X}$ (if no such n exists, set $cores.dim_{\mathcal{X}^{\perp}}(M)$=∞), and $cores.dim_{\mathcal{X}^{\perp}}$(R) is defined as sup{$cores.dim_{\mathcal{X}^{\perp}}(M)$|$M$∈ R-$Mod$}.
Also, we denote by $\mathcal{X}^{\perp}$ the class of all $\mathcal{X}$-injective modules and $^{\perp}(\mathcal{X}^{\perp})$ the class of all $\mathcal{X}^{\perp}$-projective R -modules.

Example 2.1. Let (R, m) be a commutative Noetherian and complete local domain. Assume that the depth R ≥ 2 and $cores.dim_{\mathcal{X}^{\perp}}$(R) ≤ 1. Then R/m ⊕ E(R) is an $(\mathcal{X}^{\perp})^{\perp}$-injective module.

Definition 2.2. An R-module M is said to be Gorenstein projective (G-projective for short), if there exists an exact sequence of projective modules
P = · · · $P_1 \to P_0 \to P^0 \to P^1 \to$ · · ·

such that M ≅ Im($P_0 \to P^0$) and such that $Hom_R(-, Q)$ leaves the sequence **P** exact whenever Q is a projective module.
The exact sequence **P** is called a complete projective resolution.
The Gorenstein injective (G-injective for short) modules are defined dually.

Definition 2.3. An R-module M is said to be Gorenstein flat (G-flat for short),if there exists an exact sequence of flat modules
F = · · · $F_1 \to F_0 \to F^0 \to F^1 \to$ · · ·
such that M ≅ Im($P_0 \to P^0$) and such that _ ⊗ I leaves the sequence F exact whenever I is an injective module.
The exact sequence **F** is called a complete flat resolution.

Definition 2.4. An R-module M is called Gorenstein $\mathcal{X}$-projective, if there exists an exact sequence of $\mathcal{X}$ -projective modules
P = · · · $P_1 \to P_0 \to P^0 \to P^1 \to$ · · ·
such that M ≅ Im ($P_0 \longrightarrow P^0$) and such that $Hom_R(-, Q)$ leaves the sequence **P** exact whenever Q ∈ $\mathcal{X}$ -projective module.
The Gorenstein $\mathcal{X}$-injective modules are defined dually.

Definition 2.5. An R-module M is said to be Gorenstein $\mathcal{X}$- flat (G-flat for short),if there exists an exact sequence of $\mathcal{X}$- flat modules
F = · · · $F_1 \to F_0 \to F^0 \to F^1 \to$ · · ·
such that M ≅ Im($P_0 \to P^0$) and such that _ ⊗ I leaves the sequence F exact whenever I is an $\mathcal{X}$-injective module.
The exact sequence **F** is called a complete flat resolution.

Definition 2.6. A complete projective resolution of the form

$$\boldsymbol{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

is called strongly complete projective resolution and denoted by (**P**, f).
An R-module M is called strongly Gorenstein projective (SG-projective for short) if M ≅ Ker f for some strongly complete projective resolution (**P**, f).

The strongly Gorenstein injective (SG-injective for short) modules are defined dually.

Definition 2.7. A complete flat resolution of the form

$$\boldsymbol{F} = \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

is called strongly complete flat resolution and denoted by (**F**, f).
An R-module M is called strongly Gorenstein flat (SG-flat for short) if M ≅ Ker f for some strongly complete flat resolution (**F**, f).

3. STRONGLY GORENSTEIN $\mathcal{X}$-PROJECTIVE AND STRONGLY GORENSTEIN $\mathcal{X}$-INJECTIVE MODULES

In this section, we have introduced and examined the strongly Gorenstein $\mathcal{X}$-projective and $\mathcal{X}$-injective modules which are defined as follows:

Definition 3.1. An R-module M is called strongly Gorenstein$\mathcal{X}$-projective (SG$\mathcal{X}$-projective for short), if there exists an exact sequence of $\mathcal{X}$ -projective modules

$$\boldsymbol{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that M ≅ Ker f and such that $Hom_R(-,Q)$ leaves the sequence **P** exact whenever Q ∈$\mathcal{X}$.
The strongly Gorenstein $\mathcal{X}$ -injective (SG$\mathcal{X}$-injective for short) modules are defined dually.

Example 3.2. Let (R,m) be a commutative Noetherian and complete local domain. Assume that the depthR ≥ 2 and $cores.dim_{\mathcal{X}^\perp}$(R)≤ 1. Then there exists a SG$\mathcal{X}$-projective and SG$\mathcal{X}$-injective ideal $\overline{R/m}$.
Proof. Since every projective module is $(\mathcal{X}^\perp)^\perp$-projective module, R/m is $(\mathcal{X}^\perp)^\perp$-projective module.
Also by example 2.1, R/m is $(\mathcal{X}^\perp)^\perp$-injective module.
Then there exists an exact complete $(\mathcal{X}^\perp)^\perp$-projective and $(\mathcal{X}^\perp)^\perp$-injective resolution

$$\boldsymbol{P} = \cdots \xrightarrow{f} \mathrm{R/m} \xrightarrow{f} \mathrm{R/m} \xrightarrow{f} \mathrm{R/m} \xrightarrow{f} \cdots$$

such that Ker f R/m (say).
Clearly, $Hom_R(-,Q)$ and $Hom_R(Q,-)$ leaves the sequence **P** exact whenever
Q ∈$\mathcal{X}$⊆$(\mathcal{X}^\perp)^\perp$
This implies that R/m is SG$(\mathcal{X}^\perp)^\perp$-projective and SG$(\mathcal{X}^\perp)^\perp$-injective.

Proposition 3.3. (1) If $(P_i)_{i\in \mathrm{I}}$ is a family of strongly Gorenstein $\mathcal{X}$-projective modules, then $\bigoplus P_i$ is strongly Gorenstein $\mathcal{X}$-projective.
(2) If $(I_i)_{i\in \mathrm{I}}$ is a family of strongly Gorenstein $\mathcal{X}$-injective modules, then ΠI_i is strongly Gorenstein $\mathcal{X}$-injective.

Proof. We know that a sum (resp., product) of strongly complete $\mathcal{X}$–projective (resp., $\mathcal{X}$-injective) resolutions is also a strongly complete $\mathcal{X}$ -projective (resp., $\mathcal{X}$ -injective) resolution [11, theorem 2.4].
Then $\bigoplus P_i$ is strongly Gorenstein $\mathcal{X}$-projective and ΠI_i is strongly Gorenstein $\mathcal{X}$-injective modules.

Remark 3.4. If we want to construct an example of a non-finitely generated strongly Gorenstein $\mathcal{X}$-projective module, we can see easily, from the previous Proposition and using the ideal (R/m) of example 3.2, that the direct sum (R/m)(I) for any infinite index set I is a non-finitely generated strongly Gorenstein $\mathcal{X}$- projective module.
It is clear that the strongly Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) modules are a particular case of the Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) modules.
Also every $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) module is Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) since there is an exact complex

$$0 \to P \xrightarrow{=} P \to 0$$

with P an $\mathcal{X}$-projective left R-module and such that $Hom_R(-,Q)$ leaves the above sequence exact whenever Q ∈$\mathcal{X}$. [6, Observation 4.2.2].

The next result shows that the class of all strongly Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) modules is between the class of all $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) modules and the class of all Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) modules.

Proposition 3.5. Every $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) module is strongly Gorenstein $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective).
Proof.
Let P be a $\mathcal{X}$-projective R-module, and consider the exact sequence

$$\boldsymbol{P} = \cdots \xrightarrow{f} P \oplus \mathrm{P} \xrightarrow{f} P \oplus \mathrm{P} \xrightarrow{f} P \oplus \mathrm{P} \xrightarrow{f} \cdots$$

$(x,y) \to (0,x)$

We have $0 \oplus P = Ker f = Im f \cong P$.
Let Q $\in\mathcal{X}$; applying the functor $Hom_R(-,Q)$ to the above sequence **P**, we get the following commutative diagram:

$$\begin{array}{ccc} \cdots \to Hom(P \oplus P, Q) & \xrightarrow{Hom_R(f,Q)} & Hom(P \oplus P, Q) \to \cdots \\ \cong\downarrow & & \cong\downarrow \\ \cdots \to Hom(P,Q) \oplus Hom(P,Q) & \xrightarrow{Hom_R(f,Q)} & Hom(P,Q) \oplus Hom(P,Q) \to \cdots \end{array}$$

Since the lower sequence in the diagram above is exact, the upper sequence also exact. Then P strongly Gorenstein $\mathcal{X}$-projective module.
Similarly, we can prove every $\mathcal{X}$-injective module is strongly Gorenstein $\mathcal{X}$-injective.

The next result gives a simple characterization of the strongly Gorenstein $\mathcal{X}$-projective modules.

Proposition 3.7. For any module M, the following are equivalent:
(1) M is strongly Gorenstein $\mathcal{X}$-projective;
(2) there exists a short exact sequence 0 → M → P → M → 0, where P is a $\mathcal{X}$ -projective module, and $Ext_R^1(M,Q) = 0$
for any Q $\in\mathcal{X}$;
(3) there exists a short exact sequence 0 → M → P → M → 0, where P is a $\mathcal{X}$-projective module, and $Ext_R^1(M,Q') = 0$ for any module Q' with finite $\mathcal{X}$-dimension and $cores.dim_{\mathcal{X}}(M)$ =1;`
(4) there exists a short exact sequence 0 → M → P → M → 0, where P is a $\mathcal{X}$ -projective module; such that for any Q $\in\mathcal{X}$, the short sequence 0 → Hom(M,Q) → Hom(P,Q) → Hom(M,Q) → 0 is exact;
(5) there exists a short exact sequence 0 → M → P → M → 0, where P is a $\mathcal{X}$ -projective module, such that for any module Q' with finite $\mathcal{X}$-dimension and
$cores.dim_{\mathcal{X}}(M)$ =1, the short sequence 0 → Hom(M, Q') →Hom(M, Q') →Hom(M, Q') → 0 is exact.`
Proof. (2) ⇔ (3). Let Q'be any module with finite $\mathcal{X}$-dimension, say n and let $cores.dim_{\mathcal{X}}(M)$ =1
Then we have an exact sequence

$$0 \to Q_n \cdots \to Q_1 \to Q_0 \to Q' \to 0$$

Applying the functor Hom(M,−) to the above exact sequence, we get, $Ext_R^1(M,Q') = 0$
Conversely, if Q $\in\mathcal{X}$, then $\mathcal{X}$ -dimension of Q is 0. Then by (3), $Ext_R^1(M,Q) = 0$.
The other implications follows from the definition of Ext and strongly Gorenstein $\mathcal{X}$-projective module.

4. STRONGLY GORENSTEIN $\mathcal{X}$-FLAT MODULES

In this section, we have introduced and studied the strongly Gorenstein $\mathcal{X}$-flat modules,and further we have linked them with the strongly Gorenstein $\mathcal{X}$-projective modules.
Definition 4.1. An R-module M is called strongly Gorenstein $\mathcal{X}$ -flat (SG $\mathcal{X}$ -flat for short), if there exists an exact sequence of $\mathcal{X}$-flat modules

$$\boldsymbol{F} = \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that M ≅ Ker f and such that $Hom_R(-,Q)$ leaves the sequence $\boldsymbol{F}$ exact whenever $Q \in \mathcal{X}$.
Definition 4.2. A complete flat resolution of the form is called a strongly complete flat resolution and denoted by (**F**, f). An R-module M is called strongly Gorenstein $\mathcal{X}$ - flat (SG $\mathcal{X}$ -flat for short) if M ≅ Ker f for some strongly complete flat resolution (**F**, f).

Consequently, the strongly Gorenstein $\mathcal{X}$ -flat modules are simple particular cases of Gorenstein $\mathcal{X}$ -flat modules.
Proposition 4.3. Every $\mathcal{X}$-flat module is strongly Gorenstein $\mathcal{X}$-flat.

Proposition 4.4. Every direct sum of strongly Gorenstein $\mathcal{X}$-flat modules is also strongly Gorenstein $\mathcal{X}$ -flat.
Proof. Immediate from the proof of Proposition 3.3 using the fact that tensor products commute with sum

Also, similarly to Proposition 3.7, we have the following characterization of thestrongly Gorenstein $\mathcal{X}$

-flat modules

Proposition 4.5. For any module M, the following are equivalent:
(1) M is strongly Gorenstein $\mathcal{X}$-flat;
(2) there exists a short exact sequence 0 → M → F → M → 0, where F is a $\mathcal{X}$ -flat module, and $Tor(M, I) = 0$ for any $\mathcal{X}$ -injective module I;
(3) there exists a short exact sequence 0 → M → F → M → 0, where F is a $\mathcal{X}$-flat module, and $Tor(M, I') = 0$ for any module I' with finite injective dimension;
(4) there exists a short exact sequence 0 → M → F → M → 0, where F is a $\mathcal{X}$ -flat module; such that for any $\mathcal{X}$ -injective module I, the short sequence 0 → M⊗ I → F⊗ I → M⊗ I → 0, is exact;
(5) there exists a short exact sequence 0 → M → F → M → 0, where F is a $\mathcal{X}$ -flat module, such that for any module I' with finite projective dimension, the short sequence 0 → M⊗ I′ → F⊗ I′ → M⊗ I′ → 0, is exact

Proposition 4.6. A strongly Gorenstein $\mathcal{X}$ -flat module is $\mathcal{X}$ -flat if, and only if, it has finite flat dimension.
Proof. Immediate from Proposition 3.7.

Corollary 4.7. If R has finite weak global dimension. Then, an R-module is Gorenstein $\mathcal{X}$-flat if, and only if, it is $\mathcal{X}$ –flat.

Proposition 4.8. A module is finitely generated strongly Gorenstein $\mathcal{X}$ -projective if, and only if, it is finitely presented strongly Gorenstein $\mathcal{X}$ -flat.

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