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Generating Function Of Extended Jacobi Polynomial Involving Stirling Number of Second Kind

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***Abstract*:**

In this paper we have obtained the generating function of extended Jacobi polynomial containing Stirling number of the second kind. Some particular cases are also obtained

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1) Introduction :-

With a view to unify the study of classical orthogonal polynomials (Jacobi, Laguerre and Hermite) and Bessel polynomials. Thakare [7,8] introduced the extended Jacobi polynomials denoted by Fn(α, β;x) that are defined by the following Rodrigues formula

$$F_n(\propto,\beta;x)=\frac{1}{K_n\omega(x)}D^n[\omega(x)\{X(x)\}] \qquad \text{..............(1.1)}$$

Where n=0,1,2………
Where the weight function $\omega(x)$,the quadratic function X(x) and the constants Kn are given by

$$\omega(x)=\frac{(x-a)^{\alpha}(b-x)^{\beta}}{(b-a)^{\alpha+\beta+1}\,\beta(\alpha+1,\beta+1)}\dots \qquad \text{....................(1.2)}$$

Where $\alpha>-1,\beta>-1$
$X(x)=c(x-a)(b-x), with\ c>0\dots$ ....(1.3)

$\&\ K_n=(-1)^n$n! ....(1.4)

Thakare [7] obtained unification of Orthogonal Polynomials namely Jacobi polynomials , Laguerre polynomials and Bessel polynomials by defining the polynomial

$$F_n(\propto,\beta;x)=\sum_{k=0}^{n}\binom{n+k}{k}\binom{\beta+n}{n-k}\left[\frac{(x-a)\lambda}{(b-a)}\right]^k\left[\frac{(x-a)\lambda}{(b-a)}\right]^{n-k} \qquad \text{....................(1.5)}$$

These polynomials are called as extended Jacobi polynomials and it is related to classical orthogonal polynomials as

i) Jacobi polynomials:- when -a=b=λ=1 we have

$F_n(\propto,\beta;x) = P_n^{(\beta,\lambda)}(x)$ ……………………………………(1.6)

(ii) Laguerre polynomials :- when a=0, β=b, λ=1 we have

$\lim_{b\to\infty} F_n(\propto,b;x) = (-1)^n L_n^{\alpha}(x)$ …………………………………(1.7)

(iii) Hermite polynomials: - when , $\alpha = \beta$, -a = b=$\sqrt{\alpha}$, (α>0)and in view of λ ⟶ $\frac{2}{\sqrt{\lambda}}$ (see Thakare [7]) we get

$\lim_{\alpha\to\infty} F_n(\alpha,\alpha;x)=\frac{H_n(x)}{n!}$ ....(1.8)

Following is more general generating function for the polynomials Fn (α, β;x) obtained by Patil [3]

$$\sum_{n=0}^{\infty}\binom{k+j}{j}F_{k+j}\,(\alpha-j,\beta-j;x)t^j = \left[1-\frac{\lambda t(b-x)}{(b-a)}\right]^{\alpha}\left[1+\frac{\lambda t(b-x)}{(b-a)}\right]^{\beta}$$

$$F_k\left(\alpha,\beta;x-\frac{\lambda t(x-a)(b-x)}{(b-a)}\right) \quad \text{…………………………………(1.9)}$$

Recently Gabutti and Lyness [1] ,Mathis and Sismondi [2] and Srivastava [5] obtained families of generating functions associated with the Stirling numbers S(n,k) defined by

$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$ ……..……………(1.10)

So that

$S(n,0) = 1\,,\qquad for\; n = 0$

= 0, for n in N . …………………………(1.11)

& $S(n,1) = S(n,n)\; and$

$S(n,n-1) = \binom{n}{2}$ ………………………..(1.12)

The purpose of this paper is to obtain the generating function of extended Jacobi polynomial containing Stirling number of the second kind by using the method of Srivastava [5] . Generating Function of Extended Jacobi Polynomials, Hermite polynomials and Laguerre polynomials associated with stirling numbers of second kind are obtained as particular cases.

2) Generating Function :

Replacing t by '–z 'in equation (1.9), multiplying by $K^n Z^k$ on both sides then summing from k=0 to k=∞, we get

$$\sum_{k=o}^{\infty}\sum_{j=o}^{\infty}(-1)^j\binom{k+j}{j}F_{k+j}(\alpha-j,\beta-j;x)z^{k+j}k^n$$

$$= \sum_{k=0}^{n}\left[1+\frac{\lambda z(b-x)}{(b-a)}\right]^{\alpha}\left[1-\frac{\lambda z(x-b)}{(b-a)}\right]^{\beta} \quad k^n z^k F_k\left(\alpha,\beta;x+\frac{\lambda z(x-a)(b-x)}{(b-a)}\right)$$

………………………(2.1)

Using the identity for double summation

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k)$$

………………(2.2)

See Srivastava [6 , P-100] ,equation (2.1) reduces to

$$\sum_{j=o}^{\infty}\sum_{k=o}^{j}\binom{j}{j-k} F_j(\alpha-j+k,\beta-j+k;x)k^n(-1)^{j-k} z^j$$

$$= \sum_{k=0}^{\infty}\left[1+\frac{\lambda z(b-x)}{(b-a)}\right]^{\alpha}\left[1-\frac{\lambda z(x-a)}{(b-a)}\right]^{\beta} \quad k^n z^k F_k\left(\alpha,\beta;x+\frac{\lambda z(x-a)(b-x)}{(b-a)}\right)$$

…………………..(2.3)

Replace α= α-k and β= β-k in equation (2.3) we get

$$\left[1+\frac{\lambda z(b-x)}{(b-a)}\right]^{\alpha}\left[1-\frac{\lambda z(x-a)}{(b-a)}\right]^{\beta}\sum_{k=0}^{\infty} k^n F_k\left(\alpha-k,\beta-k;x+\frac{\lambda z(x-a)(b-x)}{(b-a)}\right)\left[\frac{z}{(1+\frac{\lambda z(b-x)}{(b-a)})(1-\frac{\lambda z(x-a)}{(b-a)})}\right]^k$$

$$= \sum_{j=o}^{\infty}\sum_{k=o}^{j}\binom{j}{j-k} F_j(\alpha-j,\beta-j;x)k^n(-1)^{j-k} z^j$$

……………………………..(2.4)

In the light of definition of Stirling numbers of second kind (1.10) the equation (2.4) gives the following generating function for extended Jacobi polynomials .

$$\sum_{k=o}^{\infty} k^n F_k\left(\alpha-k,\beta-k;x+\frac{\lambda z(x-a)(b-x)}{(b-a)}\right)\left[\frac{z}{\left(1+\frac{\lambda z(b-x)}{(b-a)}\right)\left(1-\frac{\lambda z(x-a)}{(b-a)}\right)}\right]^k$$

$$= \left[1+\frac{\lambda z(b-x)}{(b-a)}\right]^{-\alpha}\left[1-\frac{\lambda z(x-a)}{(b-a)}\right]^{-\beta} \sum_{k=o}^{n} k!\, S(n,k)F_k(\alpha-k,\beta-k;x)z^k$$

…………………………….(2.5)

The generating function (2.5) is also obtained by applying following Srivastava’s theorem [5].

Theorem:- Let the sequence $\{Fn(x)\}_0^{\infty}$be generated by

$$\sum_{n=0}^{\infty}\binom{n+k}{k} F_{n+k}(x)t^k = f(x,t)(g(x,t))^{-n}F_n(h(x,t)) \quad (n\in N_o)$$

…………………………….(2.6)

Where f, g and h are suitable functions of x & t .Then in terms of the Stirling numbers S(n,k) defined by (1.10) the following family of generating functions holds true

$$\sum_{n=0}^{\infty} k^n F_n\left(h(x,-z)\right)\left(\frac{z}{g(x,-z)}\right)^k = \{f(x,-z)\}^{-1}\sum_{k=o}^{n} k!\ S(n,k)F_k(x)\, z^k \qquad (n\in \mathbb{N}_0)$$

……………………………….(2.7)

Provided that each member of above equation exists.

3.Particular cases:--

(i) Jacobi polynomials –

Putting-a=b=λ=1 and using the relation (1.6) generating function (2.5) reduces to

$$\sum_{k=0}^{\infty} k^n P_k^{(\beta-k,\alpha-k)}\left(x-\frac{1}{2}(x^2-1)z\right)\left[\frac{z}{(1-\frac{z(x-1)}{2})(1-\frac{z(x+1)}{2})}\right]^k$$

$$=\left[1-\frac{1}{2}(x-1)z\right]^{-\alpha}\left[1-\frac{1}{2}(x+1)z\right]^{-\beta} \sum_{k=o}^{n} k!\ S(n,k)P_k^{(\beta-k,\alpha-k)}(x)\, z^k$$

……………………………..(2.8)

Replace ‘ α ‘ by ‘β’ and ‘β’ by ‘ α’ in equation (4.2.8) we obtain generating function of Jacobi polynomial obtained by Srivastava [5 ,equation (3.6) ,P-756]

$$\sum_{k=0}^{\infty} k^n P_k^{(\alpha-k,\beta-k)}\left(x-\frac{1}{2}(x^2-1)z\right)\left[\frac{z}{(1-\frac{z(x-1)}{2})(1-\frac{z(x+1)}{2})}\right]^k$$
$$= \left[1-\tfrac{1}{2}(x+1)z\right]^{-\alpha}\left[1-\tfrac{1}{2}(x-1)z\right]^{-\beta} \sum_{k=o}^{n} k!\ S(n,k) P_k^{(\alpha-k,\beta-k)}(x)\, z^k$$
…………………………….(.2.9)

(ii) Laguerre Polynomials:-
Putting a=0, β=b and λ=1 and using the relation (1.7) equation (2.5) reduces to

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}\left[\left(x(1+z)\right)\right]\left[\frac{-z}{1+z}\right]^k = (1+z)^{-\alpha} e^{zx} \sum_{k=o}^{\infty} k!\ S(n,k) L_k^{(\alpha-k)}(x)(-z)^k$$
…………………... (2.10)

Now replacing ‘ Z ‘ by ‘-Z ‘ in equation (2.10) we have

$$\sum_{k=0}^{\infty} k^n L_k^{(\alpha-k)}\left[\left(x(1-z)\right)\right]\left[\frac{z}{1-z}\right]^k = (1-z)^{-\alpha} e^{-zx} \sum_{k=o}^{\infty} k!\ S(n,k) L_k^{(\alpha-k)}(x)(z)^k$$
…………………(2.11)

Which is the generating function of Laguerre polynomials obtained by Srivastava [5 ,equation (3.27) ,P-760]

(iii) Hermite polynomial :-
Putting $\beta = \alpha$, -a = b =$\sqrt{\alpha}$, (α>0) in view of $\lambda \longrightarrow \frac{2}{\sqrt{x}}$ and using the relation (1.8) , equation (2.5) reduces to generating function for the classical Hermite polynomial obtained by Shy- Der Lin –Shih Tong Tu , See H.M.Srivastava [4,P-203]

$$\sum_{k=0}^{\infty} k^n \frac{H_k(x+z)}{k!} z^k = \exp(2xz+z^2) \sum_{k=0}^{n} S(n,k)\ H_k(x)(z)^k$$
……………………………….(2.12)

The generating function (2.12) for x tending to x-z assumes the form

$$\sum_{k=0}^{\infty} k^n \frac{H_k(x)}{k!} z^k = \exp(2xz-z^2) \sum_{k=0}^{n} S(n,k)\ H_k(x-z)(z)^k \qquad (n\in \mathbb{N}_0)$$
……………………………(2.13)

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