Journal of Advanced Zoology
ISSN: 0253-7214
Volume 44 Issue S-8 Year 2024 Page 279-293

Bifurcation Analysis Of Three Species Prey-Predator Delayed Food Chain Model With Holling Type-II Functional Response And Fear Effect

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Abstract

In this study, we establish and investigate a delayed three-species food chain model with Holling type-II functional response and fear impact in the prey population. We assume that the fear of predators is restricting the growth rate of the prey population. The middle predator is thought to be both a prey and a predator, whereas the top predator consumes both the prey and the middle predator. We investigate the presence of all positive equilibrium points as well as their local stability. In addition, for the non-delayed model, we perform the Hopf-bifurcation analysis around the interior equilibrium point based on the fear parameter. Furthermore, for the related delayed model, we demonstrate the lo-cal stability and presence of Hopf-bifurcation. Finally, some numerical results have been provided to validate our analytical conclusions.

Keywords: Fear effect, Holling type-II functional response, Hopf-bifurcation, Local stability, Time delay.

1. Introduction

The relationship between prey and predator is one of the most important factors in eco-logical systems, which shapes social organization and environmental sustainability. There are two techniques for capturing the predator's influence on the prey population.First,thereispreyeating(directeffect)bypredators[14],which is very easy to see in the field and has been the primary focus of mathematical ecology thus far. Other factors include fear of predators (indirect effect) on the prey population, which can alter the population of prey. There is growing evidence that fear of predators has a greater impact than direct consumption, and it plays an important role in the dynamics of predator-prey interactions. When the predator signal (chemical or vocal) is detected, the prey population frequently spends more time being attentive and less time foraging[6]. They also move from greater predation risk regions to reduced predation risk areas for forage [13, 18] and give up their higher feeding zone. Fear causes such behavioral changes, which can cause physiological stress in prey species and have a severe impact on their reproduction methods and long-term survival. It is widely known in the literature that prey populations for ageless due to predator fear. Cannon, W. B., [1] was the first to describe the fear factor. Prey individuals are always scared to step out into the open habitat because of the fear created by predators, and they don't have any free environment for everyday activities such as breeding. As a result, the prey population reproduction rate is suffers by their fear of predators. As the above fact, the cost of fear must be seen as a form of reproductive decline. Fear is a significant aspect that must be investigated first in the field of the environment. Several research studies have been conducted to investigate how fear affects species. [11] They investigated the effect of the cost of fear on the sustainability of bifurcating periodic solutions. They discovered that fear generates a number of limit cycles. [10]They analyzed a modified Leslie-Gower predator-prey model that included hunting cooperation and a fear effect. They investigated the model with and without fear effect. They have concluded that, in the absence of the fear effect, hunting cooperation can induce Hopf-bifurcation. Furthermore, they observed that the fear factor can stabilise the model by avoiding the occurrence of periodic solutions and making the system very robust compared to hunting cooperation. [20] They examined the impact of fear effect with the Holling-Type-II predator-prey model, which includes prey refuge. As a result of the analysis, they concluded that the effect of fear could not only reduce the population density of predators in a positive equilibrium but also stabilize the system by avoiding the occurrence of periodic solutions. The first mathematical model with an effect of fear was proposed by[17], they analyzed the proposed model by including the cost of fear to the prey population. They observed the stabilizing effects of fear. The most important finding was that, for the right combination of parameter values, the influence of fear would alter the stability attribute of the limit cycle oscillations. Recently, many authors studied the prey-predator model with fear effect, delay, prey refuge and different functional responses[2,4,5,8,16].

In general, time delays can affect the stability of prey-predator models, since there is a time delay for every biological occurrence. In nature, the delayed models are much more realistic. The behavior of delayed differential equations are much more complicated than that of ordinary differential equations. The effect causes prey to be destroyed inside the predatory population, which also does not emerge immediately in the prey predator system. As a result, there is a time delay, which is known as gestational delay.[19]the authors investigated the dynamical behavior of Holling type-II three species food chain model with delay. They provided the conditions for existence of local stability and Hopf-bifurcation. They also provided the conditions for the direction of Hopf-bifurcation for the proposed model. They concluded that the delay plays an important role in food chain model. [19]the authors investigated the dynamical behavior of Holling type-II three species food chain model with delay. They provided the conditions for existence of local stability and Hopf-bifurcation. They also provided the conditions for the direction of Hopf-bifurcation for the proposed model. [9]the authors extensively investigated the dynamics of the interior equilibrium with a delay, and they explored how the gestation delay might cause chaos. They discovered that increasing the value of delay causes the system to lose stability and that limit cycle oscillations occur. The authors in [12] analyzed the prey-predator model with fear effect and delay. They observed that by varying the delay parameter over a Hopf-bifurcation point, the coexisting equilibrium point can transition between a stable steady state and stable limit cycles. They also discovered that when the time delay is very high, predator and prey may coexist and produce chaotic oscillations in the delay system. [7]the authors discussed the local, global stability and Hopf-bifurcation analysis for the intraguild predation model with time delay. They observed that the delay causes the stability of the equilibrium point. In[15] the authors discussed the dynamics of the food chain model by incorporating the Allee effect and delay. They observed that by increasing the level of delay, the proposed model shifted from periodic oscillation to stable. In[8] the authors explored the impact of fear effects and gestation delay in the three-species food chain model with Holling type II functional response. As per the analysis, they concluded that the cost of fear switches the stability of the non-delayed model. whereas they analyzed the delayed model, where they observed the proposed model changing from stable to a limit cycle around the interior equilibrium point. Based on the above facts, in this work we investigate the three-species food chain model, which includes fear effects in the prey population, Holling type-II functional response, and time delay in the prey population. No one has explored the proposed model with the above factors. As a result of this, we developed two mathematical models(one non-delayed and one delayed).

This paper has been reported as follows: In the next section, we study the positiveness and boundedness of solutions of the proposed model. In section 3, we discuss the Kolmogorov analysis and also investigate the existence, local stability, and occurrence of Hopf-bifurcation by choosing prey refuge parameter m as the bifurcation parameter at positive equilibrium points for the non-delayed model. In section3, we discuss the stability and existence of Hopf-bifurcation for the delayed model. Finally, this work ends with numerical simulations and a conclusion.

2. Model without time delay

In this section, we consider the three species food-chain model and the following assumptions are taken.

Parameters	BiologicalMeaning
r_1&r_2	Intrinsic growth rates of $x(t)$ and $y(t)$
γ	The consumption rate of $x(t)$
η	The predation rate of $y(t)$
δ	The commensal coefficient of $x(t)$
α&β	The natural death rate and competition ratio of $x(t)$
c_1	The half-saturation positive constant
c_2	The death rate of $z(t)$
c_3	The predation rate of $z(t)$
k	Level of fear in $x(t)$

The term $G(y,k)=\frac{1}{1+ky}$ and $\frac{y(t)}{c_1+y(t)}$ be the fear function and the Holling type – II functional response respectively.
The factor $G(y,k)$, meets the following conditions [17]

$$G(y,0)=1, G(0,k)=1, \lim_{k\to\infty} G(y,k)=0, \lim_{y\to\infty} G(y,k)=0,$$

$$\frac{\partial G(y,k)}{\partial k}<0, \qquad \frac{\partial G(y,k)}{\partial y}<0.$$

By the above mentioned assumptions, we modify the model explored in[8] with the effect of fear in prey population and Holling type-II functional response as follows:

$$\frac{dx}{dt}=\frac{r_1\,x}{1+ky}-\alpha x-\beta x^2-\gamma yx-\delta_1\,xz,$$

$$\frac{dy}{dt}=r_2 y(1-y)+\eta xy-\frac{yz}{c_1+y}, \tag{1}$$

$$\frac{dz}{dt}=z\left(-c_2+\frac{c_3 y}{c_1+y}+\delta_2 x\right)$$

with initial conditions $x(0)=x_0\geq 0, y(0)=y_0\geq 0$,and$z(0)=z_0\geq 0$.

3. Positivity and boundedness of solutions

In this section, we discuss the positive invariant and boundedness solution for the model(1).The positivity of the solution demonstrates the existence of the species, and the boundedness property places a constraint on the species' capability to grow exponentially.

$$x(t)=x(0)\exp\left(\int_0^t\left[\frac{r_1}{1+ky}-\alpha-\beta x-\gamma y-\delta_1 z\right]ds\right)\geq 0,$$

$$y(t)=y(0)\exp\left(\int_0^t\left[r_2(1-y)+\eta x-\frac{z}{c_1+y}\right]ds\right)\geq 0,$$

$$z(t)=z(0)\exp\left(\left(\int_0^t[-c_2+c_3 y/(c_1+y)+\delta_2 x]\right)\right)ds \ \geq 0.$$

Thus, the solution$\left(x(t),y(t),z(t)\right)$ with positive initial condition
$(x(0),y(0),z(0))\in R_+^3$ remains positive in the entire region R_+^3.

Theorem1. *All the solutions* $\left(x(t),y(t),z(t)\right)$ *of the model(1) with non-negative initial conditions* $(x(0),y(0),z(0))$, *which is initiate in* R_+^3 *are uniformly bounded.*

Proof. *Let us define the function*

$$Q(x(t),y(t),z(t)) = \delta_2 x(t) + \delta_1 c_3 y(t) + \delta_1 z(t).$$

Differentiate with respect to time along with(1), we have

$$\frac{dQ}{dt} = \frac{\delta r_1 x}{1+ky} - \delta_2 \alpha x - \delta_2 \beta x^2 + (\delta_1 c_3 \eta - \delta_2 \gamma)xy + \delta_1 r_2 c_3 y^2 - \delta_1 c_2 z,$$

$$\frac{dQ}{dt} + \epsilon Q \le \delta_2 r_1 x - \delta_2 \alpha x + \delta_2 \epsilon x - \delta_2 \beta x^2 + r_2 c_3 \delta_1 y + c_3 \delta_1 \epsilon y - r_2 c_3 \delta_1 y^2$$
$$-c_2 \delta_1 y^2 - c_2 \delta_1 z + \epsilon \delta_1 z,$$

[provided $\delta_1 c_3 \eta \ge \gamma \delta_2$],

$$= (r_1 \delta_2 + \delta_2 \epsilon - \delta_2 \alpha)x - \delta_2 \beta x^2 + (\delta_1 r_2 c_3 + \delta_1 c_3 \epsilon)y - \delta_1 r_2 c_3 y^2 + (\epsilon - c_2)\delta_1 z,$$

$\le \frac{\delta_2 (r_1+\epsilon-\alpha)^2}{4\beta} + \frac{\delta_1 (r_2 c_3 + c_3 \epsilon)^2}{4 r_1 c_3} = P$, where $\epsilon \le c_2$

$$\frac{dQ}{dt} + \epsilon Q \le P.$$

Implies, $0 \le Q \le \frac{P(1-e^{-\epsilon t})}{\epsilon} + Q\big(x(0), y(0), z(0)\big)e^{-\epsilon t}$.

Therefore, for $t \to \infty$ we have $0 \le Q \le \frac{P}{\epsilon}$.
Hence, all the solution of the model (1) that initiate in R^3_+ are confined in the region

$$Q = \left\{(x,y,z) \in R^3_+, Q \le \frac{P}{\epsilon} + \zeta, \forall \zeta > 0\right\}.$$

3 Analysis

The model(1) simplifies to the well-known Lotka-Volterra competition model when species z is absent. It's likely that in the absence of y, species x will expand logistically where as species z will starve to death.

3.1Kolmogorov analysis and equilibrium analysis

The Kolmogorov theorem is based on a lot of requirements, yet it can only be applied to a two-dimensional system [3]. In the absence of x, which is commensal of z species, then the model(1) reduces to Kolmogorov's model

$$\frac{dy}{dt} = r_2 y(1-y) - \frac{yz}{c_1 + y},$$

$$\frac{dz}{dt} = z\left(-c_2 + \frac{c_3 y}{c_1 + y}\right). \qquad (2)$$

Under the below Kolmogorov condition:

$$c_2 < \frac{c_3}{c_1 + 1}. \qquad (3)$$

For the Kolmogorov model (2), local stability analysis provide the below results:
1. The trivial equilibrium point $E_{00} = (0, 0)$ is exists and always unstable.
2. The equilibrium point $E_{20} = (1, 0)$ is exists and it is locally asymptotically stable if $c_2 > \frac{c_3}{c_1+1}$.
Otherwise, it will be unstable, it is clear that when the model is Kolmogorov under (3), E_{20} will always be unstable.
3. The coexistence equilibrium point $E_{23} = (\tilde{y}, \tilde{z})$ is given by

$$\tilde{y} = \frac{c_1 c_2}{c_3 - c_2},$$
$$\tilde{z} = r_2(1-\tilde{y})(c_1 + \tilde{y}).$$

The Jacobian matrix at E_{23} is

$$J(E_{23}) = \begin{pmatrix} -r_2\tilde{y} + \dfrac{\tilde{y}\tilde{z}}{(c_1+\tilde{y})^2} & -\dfrac{\tilde{y}}{c_1+\tilde{y}} \\ \dfrac{c_1 c_3 \tilde{z}}{(c_1+\tilde{y})^2} & 0 \end{pmatrix}.$$

The equilibrium point E_{23} is locally asymptotically stable, provided the following condition holds:

$$r_2 < \frac{\tilde{z}}{(c_1+\tilde{y})^2}.$$

3.2Different equilibria and their stability

In this part, we establish positive equilibrium points and then investigate the local stability of the obtained equilibrium points.

1. The trivial equilibrium point $E_0(0,0,0)$.
2. The first axial equilibrium point $E_1\left(\frac{r_1-\alpha}{\beta}, 0, 0\right)$.
3. The second axial equilibrium point $E_2(0,1,0)$.
4. The middle predator free equilibrium point $E_3\left(\frac{c_2}{\delta_2}, 0, \frac{\delta_2(r_1-\alpha)-\beta c_2}{\delta_2\delta_1}\right)$.
5. The top predator free equilibrium point $E_4(\hat{x},\hat{y},0)$, where $\hat{x} = \frac{r_2(\hat{y}-1)}{\eta}$ and $\hat{y}$ is the positive root of the following equation:

$$(\beta k r_2 + \gamma k\eta)\hat{y}^2 + (\eta k\alpha + \beta r_2 + \gamma\eta - \beta k r_2)\hat{y} + (\alpha\eta - r_1\eta - \beta r_2) = 0.$$

The equilibrium point $E_4(\hat{x}, \hat{y}, 0)$ is exist if, $\beta k r_2 < \eta k\alpha + \beta r_2 + \gamma\eta$ *and* $\alpha\eta < r_1\eta + \beta r_2$.

6. The interior equilibrium point $E^*(x^*, y^*, z^*)$, where $y^* = \frac{c_1c_2 - c_1\delta_2 x^*}{c_3 - c_2 + \delta_2 x^*}$, exist if $c_2 > \delta_2 x^*$ and $c_3 > c_2$,

$$z^* = \frac{c_1(r_2+\eta x^*)(c_3-c_2+\delta_2 x^*)^2}{c_3-c_2+\delta_2 x^*} + \frac{(r_2(1-c_1)+\eta x^*)(c_3-c_2+\delta_2 x^*)(c_1c_2-c_1\delta_2 x^*)}{(c_3-c_2+\delta_2 x^*)^2} - \frac{r_2(c_1c_2-c_1\delta_2 x^*)^2}{(c_3-c_2+\delta_2 x^*)^2},$$

and x^* be the positive root of the below equation:

$$A_1 x^{*5} + A_2 x^{*4} + A_3 x^{*3} + A_4 x^{*2} + A_5 x^* + A_6 = 0,$$

where

$A_1 = \beta c_1 \delta_2^4 k - \beta\delta_2^4,$

$A_2 = \delta_2^3\,(c_3(-4\beta + c_1(3\beta k - \delta_1\eta) + c_1^2\delta_1\eta k) - 4\beta c_2 (c_1 k - 1) + \delta_2(-\alpha + c_1(\gamma + \alpha k) + \gamma c_1^2(-k) + r_1)),$

$A_3 = \delta_2^2(6\beta c_2^2(c_1 k - 1 + c_2(-3c_3(-4\beta + c_1(3\beta k - \delta_1\eta) + c_1^2\delta_1\eta k) - 4\delta_2(-\alpha + c_1(\gamma + \alpha k) + \gamma c_1^2(-k) + r_1)) + c_3\Big(c_3(-6\beta + 3c_1(\beta k - \delta_1\eta) + 2c_1^2\delta_1\eta k) + \delta_2(-4\alpha + c_1(3(\gamma + \alpha k) - \delta_1 r_2) + c_1^2(\delta_1(k-1)r_2 - 2\gamma k) + c_1^3\delta_1 k r_2 + 4r_1)\Big)),$

$$\begin{aligned} A_4 = \delta_2\Big(&-4\beta c_2^3(c_1 k - 1) \\ &+ 3c_2^2\Big(c_3(-4\beta + c_1(3\beta k - \delta_1\eta) + c_1^2\delta_1\eta k) \\ &+ 2\delta_2(-\alpha + c_1(\gamma + \alpha k) + \gamma c_1^2(-k) + r_1)\Big) \\ &+ c_3 c_2\Big(c_3(12\beta + c_1(6\delta_1\eta - 6\beta k) - 4c_1^2\delta_1\eta k) \\ &- 3\delta_2(-4\alpha + c_1(3(\gamma + \alpha k) - \delta_1 r_2) + c_1^2(\delta_1(k-1)r_2 - 2\gamma k) + c_1^3\delta_1 k r_2 \\ &+ 4r_1)\Big) \\ &+ c_3^2\Big(c_3(-4\beta + c_1(\beta k - 3\delta_1\eta) + c_1^2\delta_1\eta k) \\ &+ \delta_2(-6\alpha + 3c_1(\gamma + \alpha k - \delta_1 r_2) - c_1^2(\gamma k - 2\delta_1(k-1)r_2) + c_1^3\delta_1 k r_2 + 6r_1)\Big)\Big), \end{aligned}$$

$$A_5 = \beta\ c_2^4(c_1\ k-1) + c_2^3(c_3(4\beta + c_1(\delta_1\eta - 3\beta\ k) + c_1^2\ \delta_1\ \eta\ (-k)) - 4\ \delta_2\ (-\alpha + c_1\ (\gamma + \alpha\ k) + \gamma\ c_1^2\ (-k) + r_1)) + c_3\ c_2^2\Big(c_3(-6\beta + 3\ c_1(\beta\ k - \delta_1\ \eta) + 2\ c_1^2\ \delta_1\ \eta\ k) + 3\ \delta_2(-4\ \alpha + c_1(3\ (\gamma + \alpha\ k) - \delta_1\ r_2) + c_1^2(\delta_1\ (k-1)r_2 - 2\ \gamma\ k) + c_1^3\ \delta_1\ k\ r_2 + 4\ r_1)\Big) - c_3^2\ c_2(c_3(-4\beta + c_1(\beta\ k - 3\ \delta_1\ \eta) + c_1^2\ \delta_1\ \eta\ k) + 2\ \delta_2(-6\ \alpha + 3\ c_1(\gamma + \alpha\ k - \delta_1\ r_2) - c_1^2(\gamma\ k - 2\ \delta_1\ (k-1)r_2) + c_1^3\ \delta_1\ k\ r_2 + 6\ r_1)) + c_3^3(\delta_2\ (-4\ \alpha + c_1(\gamma + \alpha\ k - 3\ \delta_1\ r_2) + c_1^2\ \delta_1\ (k-1)r_2 + 4\ r_1) - c_3(\beta + c_1\delta_1\ \eta)),$$

$$A_6 = (c_2 - c_3)(c_3\ c_2^2\Big(c_1\big(\delta_1\ r_2 - 2\ (\gamma + \alpha\ k)\big) + c_1^2(\gamma\ k - \delta_1\ (k-1)r_2) + c_1^3\ \delta_1\ (-k)r_2 + 3\ (\alpha - r_1)\Big) + c_3^2\ c_2(-3\ \alpha + c_1(\gamma + \alpha\ k - 2\ \delta_1\ r_2) + c_1^2\ \delta_1\ (k-1)r_2 + 3\ r_1) + c_2^3(-\alpha + c\_1\ (\gamma + \alpha\ k) + \gamma\ c_1^2\ (-k) + r_1) + c_3^3(\alpha + c_1\ \delta_1\ r_2 - r_1)).$$

Now, the Jacobian matrix of the model (1) is

$$J = \begin{pmatrix} \dfrac{r_1}{1+ky} - \alpha - 2\beta x - \gamma y - \delta_1 z & -\left(\dfrac{r_1 kx}{(1+ky)^2} + \gamma x\right) & -\delta_1 x \\ \eta y & r_2 - 2r_2 y + \eta x - \dfrac{c_1 z}{(c_1+y)^2} & -\dfrac{y}{c_1+y} \\ \delta z & \dfrac{c_1 c_3 z}{(c_1+y)^2} & -c_2 + \dfrac{c_3 y}{c_1+y} + \delta_2 x \end{pmatrix}. \quad (4)$$

Theorem 2. *The trivial equilibrium point $E_0(0, 0, 0)$ of the model (1) is always unstable.*

Proof. The characteristic equation of the model (1) at $E_0(0, 0, 0)$ is

$(r_1 - \alpha - \lambda)(r_2 - \lambda)(-c_2 - \lambda) = 0$

The eigenvalues of the above equation are $r_1 - \alpha$, r_2 and $-c_2$. Since r_2 and $-c_2$ are opposite sign, thus $E_0(0, 0, 0)$ is always unstable.

□

Theorem 3. *1. The first axial equilibrium point $E_1\left(\frac{r_1-\alpha}{\beta}, 0, 0\right)$ of the model (1) is locally asymptotically stable if $r_2 < \eta\left(\frac{\alpha-r_1}{\beta}\right)$ and unstable if $r_2 > \eta\left(\frac{\alpha-r_1}{\beta}\right)$.*

2. The second axial equilibrium point $e_2(0,1,0)$ of the model (1) is locally asymptotically stable if $r_1 < (\alpha+\gamma)(1+k)$ and $c_2(c_1+1) > c_3$.

*Proof:*1. The characteristic equation of the model (1) at $E_1\left(\frac{r_1-\alpha}{\beta}, 0, 0\right)$ is

$$(\alpha - r_1 - \lambda)\left(\left(r_2 - \eta\left(\frac{\alpha - r_1}{\beta}\right) - \lambda\right)\left(-c_2 - \delta_2\left(\frac{\alpha - r_1}{\beta}\right) - \lambda\right)\right) = 0.$$

The eigenvalues are $\lambda_1 = \alpha - r_1, \lambda_2 = r_2 - \eta\left(\frac{\alpha-r_1}{\beta}\right)$ and $\lambda_3 = -c_2 - \delta_2\left(\frac{\alpha-r_1}{\beta}\right)$.

2. The characteristic equation of the model(1) at $E_2(0,1,0)$ is

$$\left(\frac{r_1}{1+k} - \alpha - \gamma - \lambda\right)(-r_2 - \lambda)\left(-c_2 + \frac{c_3}{c_1+1} - \lambda\right) = 0.$$

The eigenvalues are $\lambda_1 = \frac{r_1}{1+k} - \alpha - \gamma, \lambda_2 - r_2$ and $\lambda_3 = -c_2 + \frac{c_3}{c_1+1}$. Thus, the second axial equilibrium point $E_2(0,1,0)$ will be locally asymptotically stable if $r_1 < (\alpha+\gamma)(1+k)$ and $c_2(c_1+1) > c_3$.

Theorem 4. *1.Middle predator free equilibrium $E_3\left(\frac{c_2}{\delta_2}, 0, \frac{\delta(r_1-\alpha)-\beta c_2}{\delta_1\delta_1}\right)$ of the model (1) is locally asymptotically stable if $N_{11} < 0, N_{22} < 0$ and $N_{13}N_{31} < 0$ where $N_{11}, N_{13}, N_{22},$ and N_{31} are given in the proof.*

2. Top predator free equilibrium $E_4(\hat{x}, \hat{y}, 0)$ of the model (1) will be locally asymptotically stable if $P_{33} < 0, P_{11} + P_{22} < 0$ and $P_{11}P_{22} - P_{12}P_{21} > 0$, where $P_{11}, P_{12}, P_{21}, P_{22}$ and P_{33} are given in the proof.

Proof. 1. The Jacobian matrix of the model (1) at $E_3\left(\frac{c_2}{\delta_2}, 0, \frac{\delta(r_1-\alpha)-\beta c_2}{\delta_1\delta_1}\right)$, which is given in the following form

$$J(E_3) = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ 0 & N_{22} & 0 \\ N_{31} & N_{32} & 0 \end{pmatrix}.$$

The characteristic equation of the above Jacobian matrix is

$$(N_{22}-\lambda)(\lambda^2-N_{11}\lambda-N_{13}N_{31})=0,$$

where

$$N_{11}=-\frac{\beta c_2}{\delta_2}, N_{12}=-\frac{r_1 k c_2}{\gamma c_2+\delta_2}, N_{13}=-\frac{\delta_1 c_2}{\delta_2},\ N_{22}=r_2-\frac{\eta c_2}{\delta_2}-\frac{1}{c_1}\left(\frac{\delta_2(r_1-\alpha)-\beta c_2}{\delta_2\delta_1}\right),$$
$$N_{31}=\frac{\delta_2(r_1-\alpha)-\beta c_2}{\delta_1}, N_{32}=\frac{c_3}{c_1}\left(\frac{\delta_2(r_1-\alpha)-\beta c_2}{\delta_2\delta_1}\right).$$

The corresponding above characteristic equation have negative real part, if $N_{22}<0$, $N_{11}<0$, and $N_{13}N_{31}<0$.

Hence, the equilibrium point $E_3\left(\frac{c_2}{\delta_2},0,\frac{\delta(r_1-\alpha)-\beta c_2}{\delta_1\delta_1}\right)$is locally asymptotically stable, if

$r_2\delta_1\delta_1 c_1<\eta c_1 c_2\delta_1+c_2(\delta_2(r_1-\alpha)-\beta c_2)$, and

$$\delta_2(r_1-\alpha)>\beta c_2.$$

2. The Jacobian matrix of the model (1) at $E_4(\hat{x},\hat{y},0)$ is given below

$$J(E_4)=\begin{pmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ 0 & 0 & P_{33}\end{pmatrix}.$$

The characteristic equation for the model (1) at $E_4(\hat{x},\hat{y},0)$ is

$$(P_{33}-\lambda)\big(\lambda^2-(P_{11}+P_{22})\lambda+(P_{11}P_{22}-P_{12}P_{21})\big)=0,$$

where

$$P_{11}-\beta\hat{x}, P_{12}=-\frac{r_1 k\hat{x}}{(1+k\hat{y})^2}-\gamma\hat{x}, P_{13}=-\delta_1\hat{x}, P_{21}=\eta\hat{y},$$
$$P_{22}=-r_2\hat{y}, P_{23}=-\frac{\hat{y}}{c_1+\hat{y}}, P_{33}=-c_2+\frac{c_3\hat{y}}{c_1+\hat{y}}+\delta_2\hat{x}.$$

The corresponding above characteristic equation have negative real part, if $P_{33}<0$, $P_{11}+P_{22}<0$ and $P_{11}P_{22}-P_{12}P_{21}>0$.

Hence, the equilibrium point $E_4(\hat{x},\hat{y},0)$ is locally asymptotically stable, if $c_2>\frac{c_3\hat{y}}{c_1+\hat{y}}+\delta_2\hat{x}$.

Theorem 5. *Interior equilibrium $E^*(x^*,y^*,z^*)$ of model (1) will be locally asymptotically stable if $\nu_1>0$, $\nu_3>0$ and $\nu_1\nu_2-\nu_3>0$, where ν_1,ν_2 and ν_3 are given in the proof.*

Proof. The Jacobian matrix at $E^*(x^*,y^*,z^*)$ is

$$J(E^*)=\begin{pmatrix}l_{11} & l_{12} & l_{13}\\ l_{21} & l_{22} & l_{23}\\ l_{31} & l_{32} & 0\end{pmatrix},$$

the characteristic equation of the above Jacobian matrix at $E^*(x^*,y^*,z^*)$ is

$$\lambda^3+\nu_1\lambda^2+\nu_2\lambda+\nu_3=0, \qquad (5)$$

where

$$\nu_1=(-l_{11}-l_{22}),$$

$$\nu_2=l_{11}l_{22}-l_{12}l_{21}-l_{23}l_{32}-l_{13}l_{31},$$

$$\nu_3=l_{11}l_{23}l_{32}-l_{12}l_{23}l_{31}-l_{13}l_{21}l_{32}+l_{13}l_{22}l_{31},$$

$$l_{11}=\beta x^*, l_{12}=-\frac{r_1 k x^*}{1+ky^*}-\gamma x^*, l_{13}=\delta_1 x^*, l_{21}=\eta y^*,$$
$$l_{22}=-r_2 y^*+\frac{y^* z^*}{(c_1+y^*)^2}, l_{23}=-\frac{y^*}{c_1+y^*}, l_{31}=\delta_2 z^*, l_{32}=\frac{c_1 c_3 z^*}{(c_1+y^*)^2}.$$

As such, if the Routh – Hurwitz condition is satisfied, the equilibrium point E^* is locally asymptotically stable: $\nu_1>0, \nu_3>0$ and $\nu_1\nu_2-\nu_3>0$.

By simple algebraic calculation we obtain $\nu_1>0$, if $-(l_{11}+l_{22})>0$,

i.e., $\beta x^*+r_2 y^*>\frac{y^*}{(c_1+y^*)^2}$. (6)

In addition, if (6) holds, then $\nu_3>0$, and we obtain the necessary condition

$$\nu_1\nu_2-\nu_3=(l_{11}+l_{22})(l_{12}l_{21}-l_{11}l_{22})+l_{12}l_{23}l_{31}+l_{13}l_{21}l_{32}>0.$$

If it satisfies $l_{12}l_{21}-l_{11}l_{22}<0$.

3.3 Hopf-bifurcation analysis

Now, we study the conditions for the existence of Hopf-bifurcation around the interior equilibrium point $E^*(x^*, y^*, z^*)$.

Theorem 6. *Model (1) possesses a Hopf-bifurcation around E^* when passes k through k^* if:*

$$i.\ \nu_1(k^*) > 0, \nu_3(k^*) > 0,$$

$$ii.\ \nu_1(k^*)\nu_2(k^*) - \nu_3(k^*) = 0.$$

For $k = k^*$ we have $\nu_1(k)\nu_2(k) = \nu_3(k)$. Then the characteristic polynomial (5) becomes

$$\left(\lambda^2 + \nu_2(k)\right)\left(\lambda + \nu_1(k)\right) = 0. \qquad (7)$$

The roots of the above equation are $\lambda_1(k) = +i\sqrt{\nu_2(k)}, \lambda_2(k) = -i\sqrt{\nu_2(k)}$ and $\lambda_3(k) = -\nu_1(k)$.
As such, for k in a neighbourhood of k^*, the roots are in the following form:
$\lambda_{1,2} = \mu(k) \pm i\eta(k)$ and $\lambda_3 = -\nu_1(k)$.
Next, we verify transversality condition

$$\left.\frac{d\left(\Re\left(\lambda_j(k)\right)\right)}{dk}\right|_{k=k^*} \neq 0, j = 1,2.$$

substituting $\lambda_j(k) = \mu(k) \pm i\eta(k)$ into (7) and differentiating with respect to k, we have

$$A(k)\mu'(k) - B(k)\eta'(k) + C(k) = 0, \qquad (8)$$

$$B(k)\mu'(k) + A(k)\eta'(k) + D(k) = 0, \qquad (9)$$

where

$$A(k) = 2\mu(k)\nu(k) + 3\mu^2(k) + \nu_2(k) - 3\eta^2(k),$$

$$B(k) = 6\mu(k)\eta(k) + 2\eta(k)\nu_1(k),$$

$$C(k) = \mu^2(k)\nu_1'(k) - \eta^2(k)\nu_1'(k) + \nu_3'(k) + \mu(k)\nu_2'(k),$$

$$D(k) = 2\mu(k)\eta(k)\nu_1'(k) + \eta(k)\nu_2'(k).$$

Multiplying equation (8) and (9) by $A(k)$ and $B(k)$, respectively, and adding those results, we get

$$A^2(k)\mu'(k) + B^2(k)\mu'(k) + A(k)C(k) + B(k)D(k) = 0. \qquad (10)$$

Substituting $\mu(k^*) = 0, \eta(k^*) = \sqrt{\nu_2(k^*)}$, then we have

$$A(k^*) = -2\nu_2(k^*), B(k^*) = 2\nu_1(k^*)\sqrt{\nu_2(k^*)},$$

$$C(k^*) = \nu_3'(k^*) - \nu_1'(k^*)\nu_2'(k^*), D(k^*) = \nu_2'(k^*)\sqrt{\nu_2(k^*)}.$$

Substituting these results in equation (10), and done the simple calculation, we have

$$\left.\frac{d\left(\Re\left(\lambda_j(k)\right)\right)}{dk}\right|_{k=k^*} = -\frac{\left(\nu_3'(k^*) - \nu_1'(k^*)\nu_2'(k^*) - \nu_1(k^*)\nu_2'(k^*)\right)}{2\left(\nu_1^2(k^*) + \nu_2^2(k^*)\right)} \neq 0, j = 1, 2,$$

if $\nu_3'(k^*) - \nu_1'(k^*)\nu_2'(k^*) - \nu_1(k^*)\nu_2'(k^*) \neq 0$ and $\lambda_3(k^*) = -\nu_1(k^*) \neq 0$.
As a result, transversality conditions hold. This indicates that Hopf-bifurcation occurs at $k = k^*$.

4 Model with time delay

In this part, we investigate the local stability and occurrence of Hopf-bifurcation for the model(1) with time delay (τ), and the proposed model takes the following form:

$$\frac{dx}{dt} = \frac{r_1\, x}{1 + ky(t-\tau)} - \alpha x - \beta x^2 - \gamma yx - \delta_1\, xz,$$

$$\frac{dy}{dt} = r_2 y(1-y) + \eta xy - \frac{yz}{c_1 + y}, \qquad (11)$$

$$\frac{dz}{dt} = z\left(-c_2 + \frac{c_3 y}{c_1 + y} + \delta_2 x\right).$$

The initial conditions are given by $x(n) = \xi_1(n) \geq 0, y(n) = \xi_2(n) \geq 0$ and $z(n) = \xi_3(n) \geq 0$ $\forall n \in [-\tau, 0]$, where, $\xi_i(n), n = 1, 2, 3$, are the continuous and bounded functions in $[\tau, 0]$. The time-delay τ is considered as a gestation time.

4.1 Local stability and Hopf-bifurcation analysis

To investigate the local stability of the model (11) around $E^*(x^*, y^*, z^*)$, use the small perturbation as follows: $\bar{x} = x(t) - x^*, \bar{y} = y(t) - y^*$ and $\bar{z} = z(t) - z^*$, where $\bar{x}, \bar{y}, \bar{z}$ are the small perturbation around x^*, y^*, z^*, respectively. Then, the linearized model of (11) at $E^*(x^*, y^*, z^*)$ is of the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 & s_{14} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(t-\tau) \\ y(t-\tau) \\ z(t-\tau) \end{pmatrix}, \qquad (12)$$

where

$$s_{11} = \beta x^*, s_{12} = -\gamma x^*, s_{13} = -\delta_1 x^*, s_{14} = -\frac{r_1 k}{(1+ky^*)^2}, s_{21} = \eta y^*,$$
$$s_{22} - r_2 y^* + \frac{y^* z^*}{(c_1+y^*)^2}, s_{23} = -\frac{y^*}{c_1+y^*}, s_{31} = \delta_2 z^*, s_{32} = \frac{c_1 z^*}{(c_1+y^*)^2}.$$

Whose characteristic equation is

$$\lambda^3 + (\zeta_1\lambda^2 + \zeta_2\lambda + \zeta_3) + (\zeta_4\lambda + \zeta_5)e^{-\lambda\tau} = 0, \qquad (13)$$

where

$\zeta_1 = -(s_{11} + s_{22}), \zeta_2 = -s_{12}s_{21} + s_{11}s_{22} - s_{23}s_{32} - s_{13}s_{31},$
$\zeta_3 = s_{11}s_{23}s_{32} - s_{12}s_{23}s_{31} - s_{13}s_{21}s_{32} + s_{13}s_{22}s_{31},$
$\zeta_4 - s_{14}s_{21}, \zeta_5 = -s_{14}s_{23}s_{31}.$

We have the following inequalities as a result of Routh-Hurwitz condition:

$\zeta_1 > 0$ and $\zeta_2 + \zeta_4 > 0$, (14)

$$\zeta_1(\zeta_2 + \zeta_4) - (\zeta_3 + \zeta_5) > 0, \qquad (15)$$
$$\zeta_1(\zeta_2 + \zeta_4) - (\zeta_3 + \zeta_5) = 0, \qquad (16)$$
$$\zeta_1(\zeta_2 + \zeta_4) - (\zeta_3 + \zeta_5) < 0. \qquad (17)$$

When $\tau = 0$, in the equation (13) we have

$$\lambda^3 + (\zeta_1\lambda^2 + \zeta_2\lambda + \zeta_3) + (\zeta_4\lambda + \zeta_5) = 0,$$

If $\lambda = iv\ (v > 0)$ in (13) we obtain

$$-iv^3 - \zeta_1 v^2 + \zeta_2 iv + \zeta_3 + (\zeta_4 iv + \zeta_5)e^{-iv\tau} = 0.$$

Separating real and imaginary part we have,

$$\zeta_1 v^2 - \zeta_3 = \zeta_4 v\, sinv\tau + \zeta_5 cosv\tau, \qquad (18)$$
$$v^3 - \zeta_2 v = -\zeta_5 sinv\tau + \zeta_4 cosv\tau. \qquad (19)$$

From, the above equations, we obtain

$$v^6 + n_1 v^4 + n_2 v^2 + n_3 = 0 \qquad (20)$$

where

$$n_1 = \zeta_1^2 - 2\zeta_2, n_2 = \zeta_2^2 - 2\zeta_1\zeta_3 - \zeta_4^2, n_3 = \zeta_3^2 - \zeta_5^2.$$

Let $\rho = v^2$, then (20) becomes

$$\rho^3 + n_1\rho^2 + n_2\rho + n_3 = 0 \qquad (21)$$

Denote $f(\rho) = \rho^3 + n_1\rho^2 + n_2\rho + n_3.$ (22)

Since, $f(0) = n_3, \lim_{\rho\to\infty} f(\rho) = +\infty$, and from (22), we have

$$f'(\rho) = 3\rho^2 + 2n_1\rho + n_2. \qquad (23)$$

Then the above equation is similar to that in [15], and we have the following lemma.

Lemma 1. *We have the following results for (21)*

1. If $n_3 \geq 0$ and $n_1^2 - 3n_2 \leq 0$, (21) has no positive root.
2. If $n_1^2 > 3n_2, 0 < \frac{-n_1+\sqrt{n_1^2-3n_2}}{3}, n_3 > 0$ and $\Delta(n_1, n_2, n_3) > 0$ or $0 < \frac{-n_1+\sqrt{n_1^2-3n_2}}{3}$, $n_3 \neq 0$ and $\Delta(n_1, n_2, n_3) > 0$: (21) has atleast two positive roots and no other roots with negative real parts, where Δ is a discriminate value of (21).

Suppose that (21) has at least one real non-negative root, and without loss of generality, we assume that (21) has three real positive roots, say ρ_1, ρ_2 and ρ_3. Then we have $v_k = \sqrt{\rho_k}$, $k = 1, 2, 3$. The corresponding threshold of time delay τ_k^i is

$$\tau_k^i = \frac{1}{v_k}\arccos\left\{\frac{(\zeta_1 v^2 - \zeta_3)\zeta_5 + \zeta_4(v^3 - \zeta_2 v)}{\zeta_5^2 + \zeta_4^2 v^2} + 2j\pi\right\},$$

where $k = 1,2,3$. $j = 0,1,2,\cdots$, and define $\tau_0 = \tau_k^0 = \min_{k=1,2,3} \tau_k^0$ and $v_0 = v_{k_0}$.

Lemma 2. *Suppose that* $\Omega_1(k)\Omega_3(k) + \Omega_2(k)\Omega_4(k) \neq 0$, *then the following transversality condition holds:* $\left[\left\{\frac{\Re d(\lambda)}{dk}\right\}\right]^{-1}_{\lambda=-iv} \neq 0$.

Proof. By taking the derivative of (13) with respect to τ, we have

$$\frac{d\lambda}{d\tau} = \frac{\left(\lambda(\zeta_4\lambda + \zeta_5)e^{-\lambda\tau}\right)}{3\lambda^2 + 2\lambda\zeta_1 + \zeta_2 + \zeta_4 e^{-\lambda\tau} - \tau(\zeta_4\lambda + \zeta_5)e^{-\lambda\tau}},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(-3v^2 + 12v\zeta_1 + \zeta_2)e^{-i\tau v} + \zeta_4}{-v^2\zeta_4 + iv\zeta_5} + \frac{i\tau}{v},$$

$$\left[\left\{\frac{\Re d\lambda}{d\tau}\right\}\right]^{-1} = \frac{\left(\Omega_1(k)\Omega_3(k) + \Omega_2(k)\Omega_4(k)\right)}{\Omega_1^2(k) + \Omega_2^2(k)} \neq 0.$$

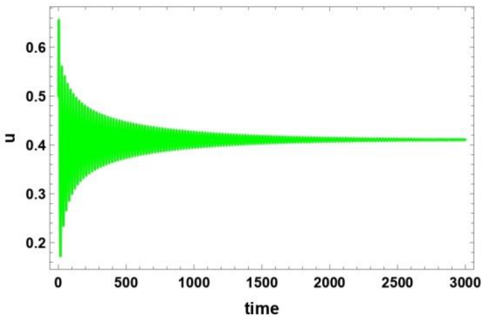
If $\Omega_1(k)\Omega_3(k) + \Omega_2(k)\Omega_4(k) \neq 0$ where

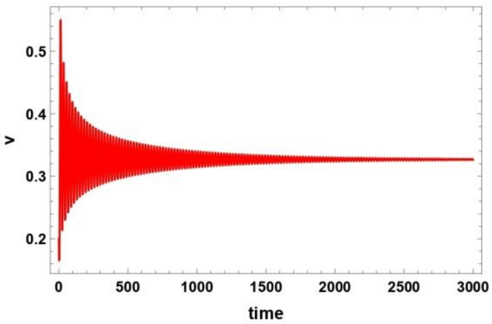
$$\Omega_1(k) = -\zeta_4 v^2,$$
$$\Omega_2(k) = -\zeta_5 \mathrm{v},$$
$$\Omega_3(k) = (\zeta_2 - 3v^2)cosv\tau - 2\zeta_1 sinv\tau + \zeta\_4,$$
$$\Omega_4(k) = \zeta_1 v\, cosv\tau + (\zeta_2 - 2v^2)sinv\tau.$$

Hence, all the conditions are satisfied. As a result, Hopf-bifurcation occurs at $\tau = \tau^*$.

5 Numerical Simulations

Numerical validation of the findings is always a necessary part of analytic investigations.In order to show the analytical findings and stability results obtained in the previous sections. We numerically simulate the solutions of the model (1) by using MATLAB ode45 and dde23 solvers for 2000-time steps, with set of parameter values $r_1 = 1.05$, $\alpha = 0.05$, $\beta = 1$, $\gamma = 1.02$, $\delta_1 = \delta_2 = 0.026$, $r_2 = 0.5$, $\eta = 0.43$, $c_1 = 0.2$, $c_2 = 0.3209$, $c_3 = 0.5$. By choose k as a bifurcation parameter to analyze the stability of model (1) around interior equilibrium point E^*. By taking the fear parameter value $k = 0.45$ then the model (1) shows the limit cycle oscillation around the interior equilibrium point $E^*(0.410655, 0.326923, 0.270379)$ which is clearly depicted in Figure (1), keep on in- creasing the value of $k = 0.95$ then the model (1) becomes a locally asymptotically stable near the equilibrium $E^*(0.53649, 0.31800, 0.29614)$ it shown in Figure (2) . Next, for the delayed model (11), we fix the delay parameter as a bifurcation parameter. Here we choose $\tau = 0.12$ and remaining parameter values are same, then the model (1) is locally asymptotically stable around the equilibrium point$E^*(0.410655, 0.326923, 0.270379)$, it displayed in the Figure (3). Furthermore, we increase the value of the delay parame- ter $\tau = 0.24$ then the model (11) losses its stability and under goes Hopf-bifurcation, which is shown in Figure (4). Also, for the better visualization of dynamical changes for the both delayed and non-delayed models (11) and (1), we plot the bifurcation diagrams respectively in Figure (5) and (6) with $k \in (0.1, 3]$ and for delay $\tau \in (0.1, 0.5]$.





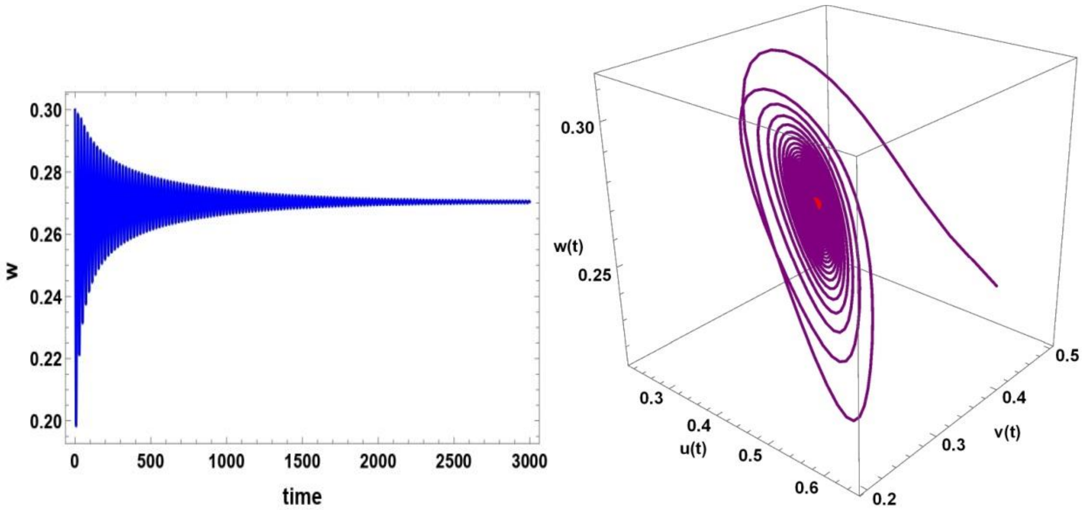


Figure 1: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (1) when k = 0.95.

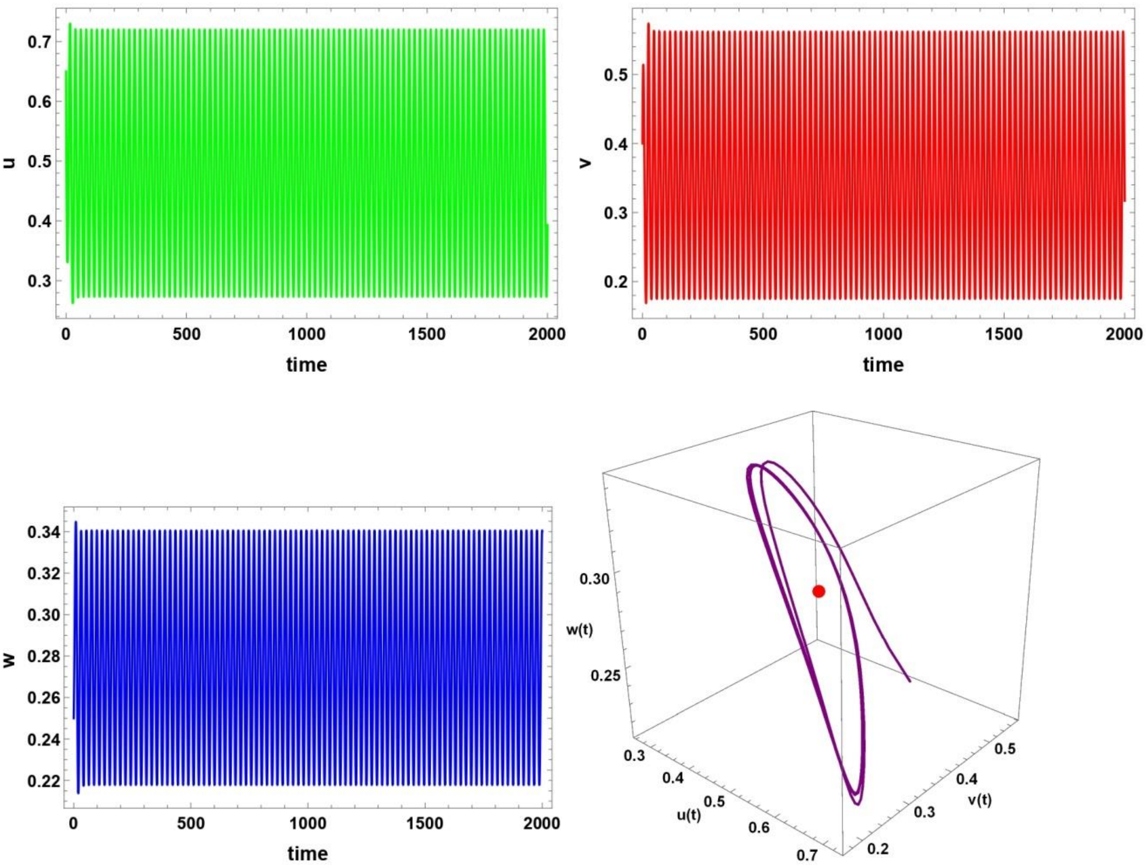


Figure 2: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (1) when k = 0.45.

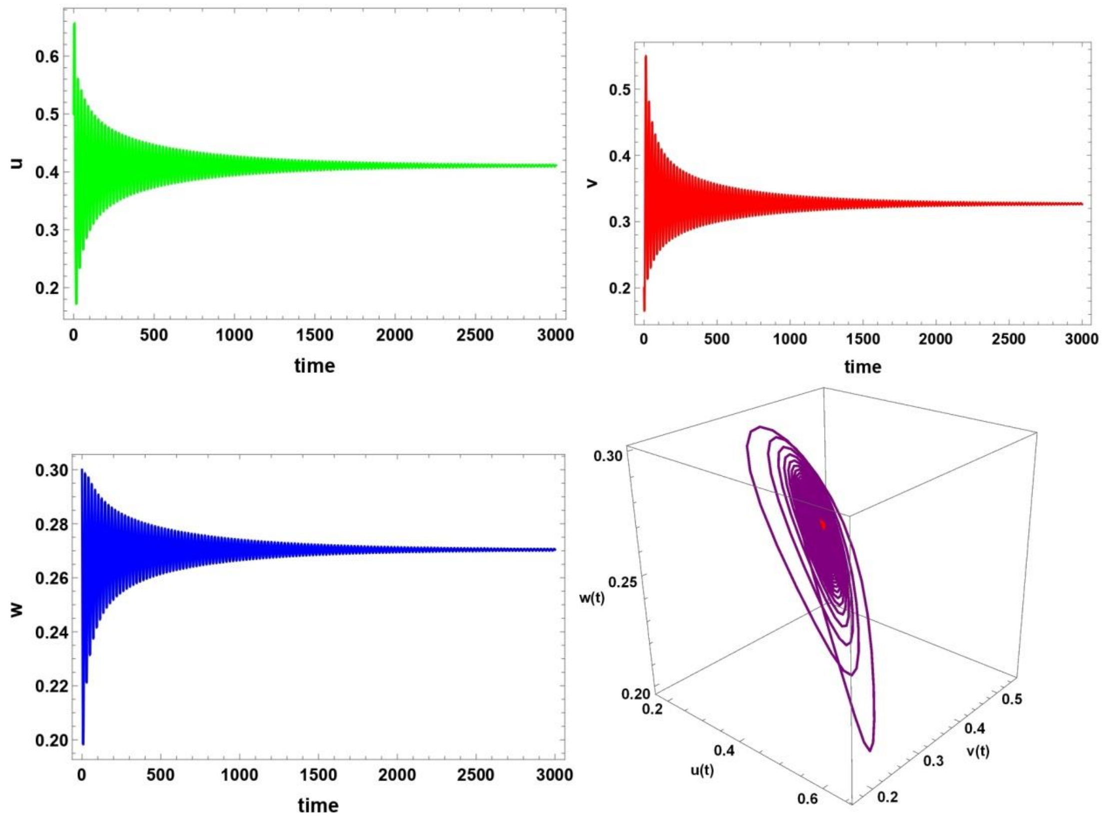


Figure 3: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (11) when τ = 0.12.

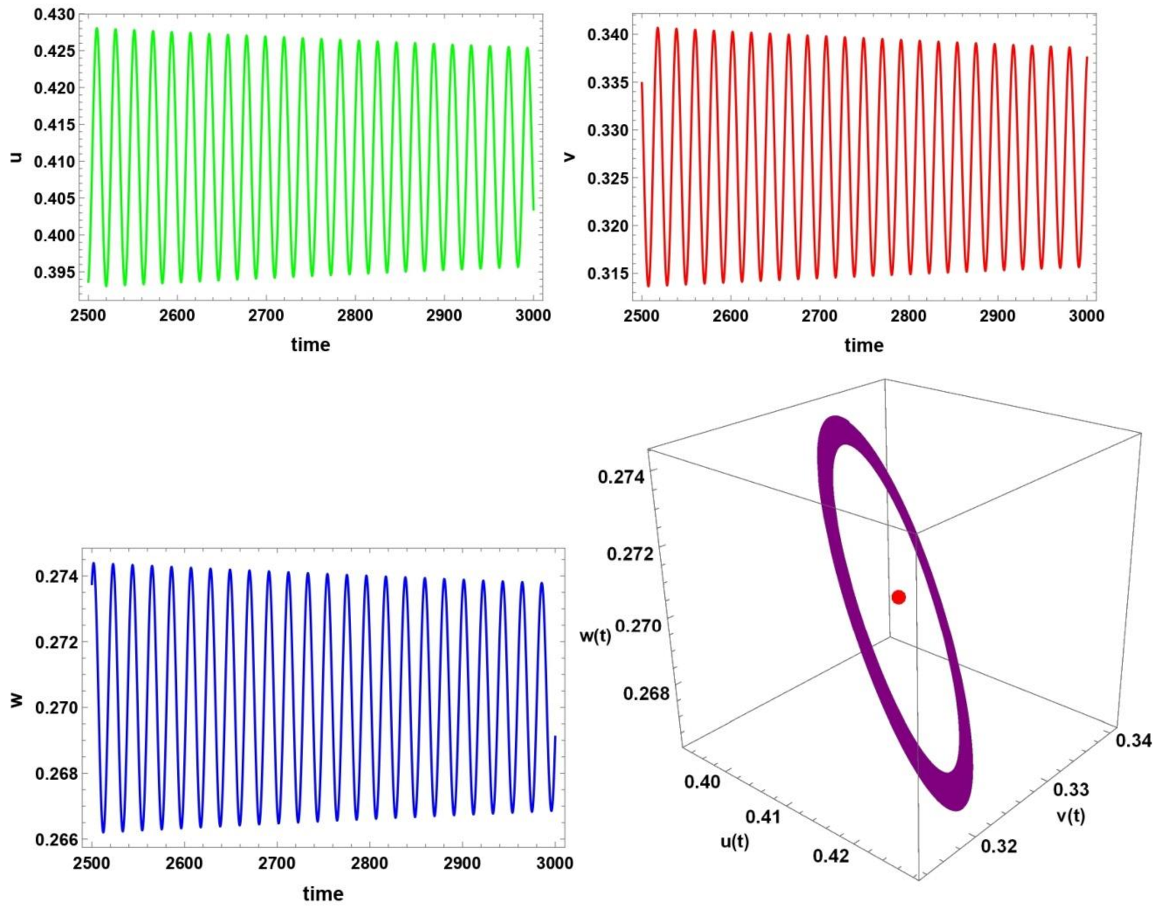


Figure 4: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (11) when τ = 0.24.

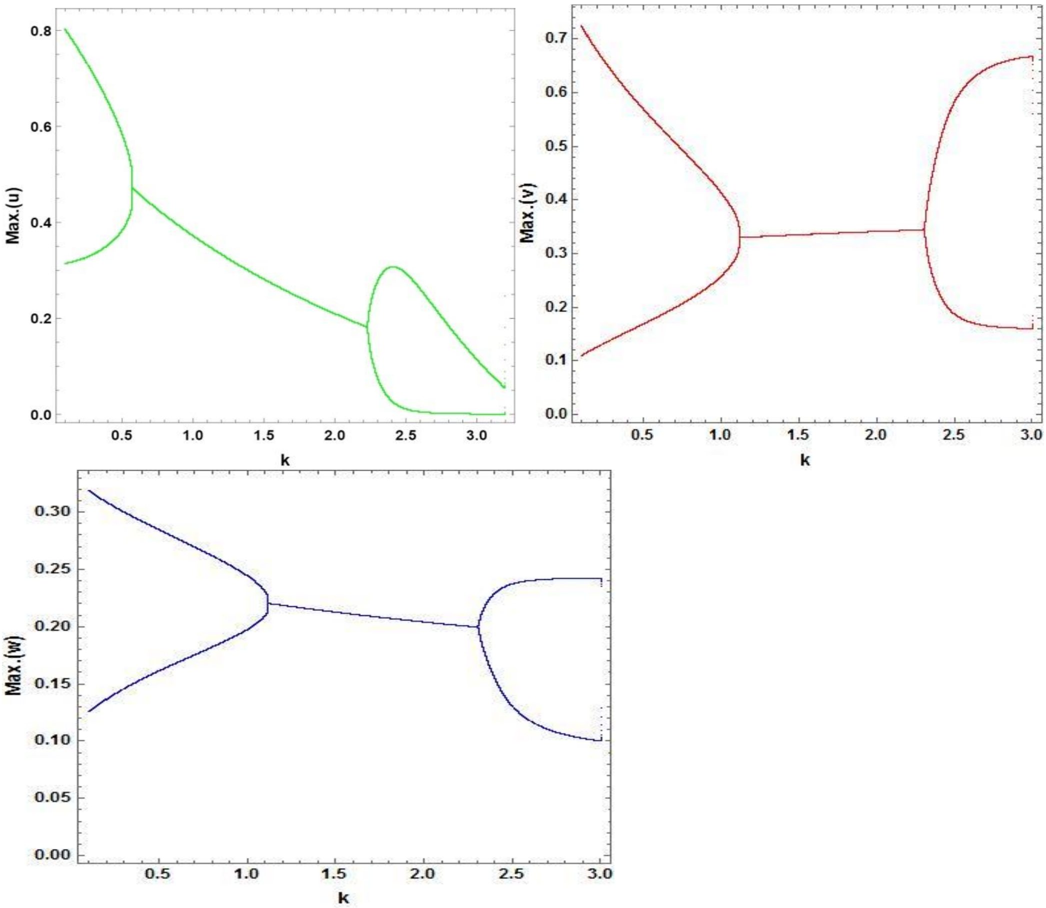


Figure 5: The bifurcation diagram for the model (1) with $k \in (0.1, 3]$.

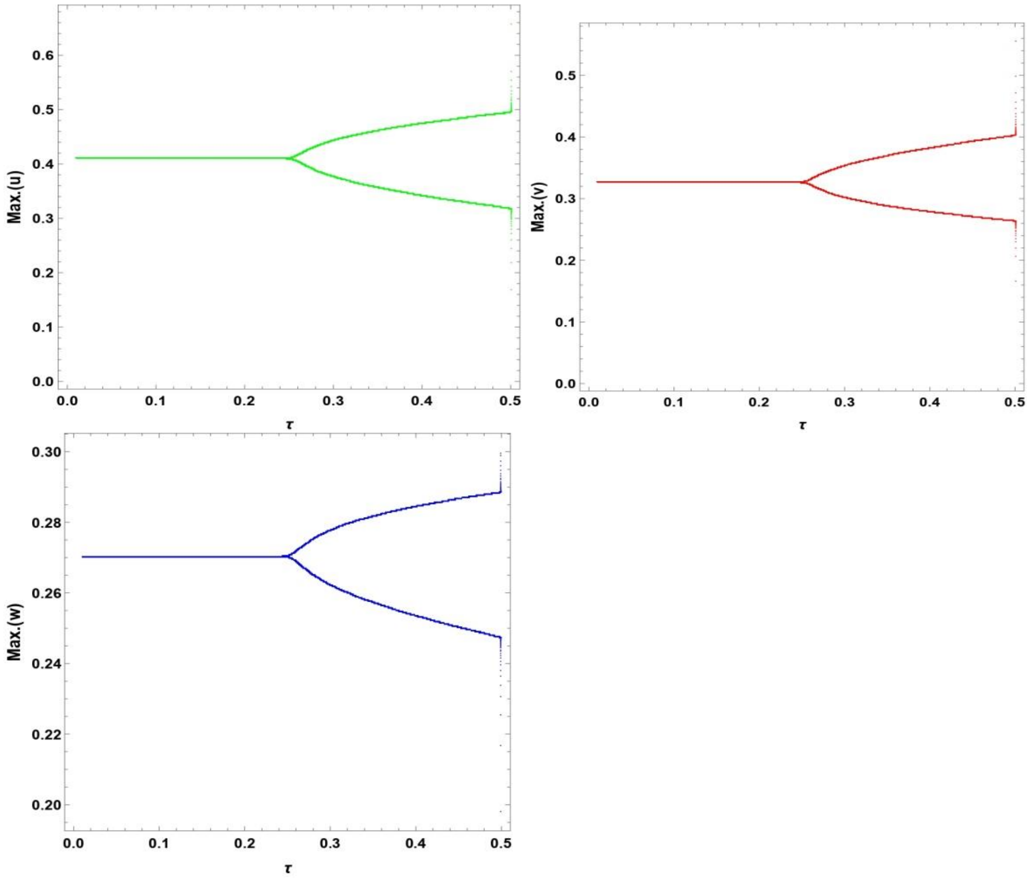


Figure 6: The bifurcation diagram for the model (11) with $k \in (0.1, 1]$ and $\tau = 0.24$.

6 Conclusion

In this chapter, we modified and investigate the food chain model which was analyzed in [8] with Holling type-II functional response. We hypothesized that when the population of prey declines due to fear of predators, so will the growth rate of the prey population. It takes some time for predators to mature from prey, and this process isn't completed instantaneously. In this fact, we incorporated the delay in the process of predator's fear term in the prey population. We want to see how the proposed model affects the prey population's cost of fear, both with and without a time delay. Analytical and numerical investigations were carried out on our model in order to meet our aims. As a result of this, we discovered a correlation between the evolution of prey species and the development of their fear of predators. To begin, we provide an analytical demonstration of the existence of positive equilibrium points. Furthermore, we investigated the stability of the suggested model (1) and discovered that changing the cost of fear k has an immediate effect on the stability of the proposed model (1) without a delay. In addition, we conducted local stability and Hopf-bifurcation evaluations of the proposed model (11) with delay. This leads to Hopf- bifurcation around E^* for increasing delays. Numerical simulations are used to verify the theoretical results. We come to the conclusion that the Hopf-bifurcation in the proposed models has a stronger impact on stability switches when fear and delay are present. The non-delayed and delayed models' time series, phase portraits, and bifurcation diagrams with respect to the fear effect and time delay are then drawn.

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