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Connected Vertex-Edge Dominating Sets and Connected Vertex-Edge Domination Polynomials of Friendship F_n

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Abstract

Let G be a simple connected graph of order n. Let $D_{cve}(G,i)$ be the family of connected vertex-edge dominating sets in G with cardinality i .The polynomial $D_{cve}(G,x) = \sum_{i=\gamma_{cve}}^{n} d_{cve}(G,i)\, x^i$ is called the connected vertex - edge domination polynomial of G, where $d_{cve}(G,i)$ is the number of connected vertex - edge dominating sets of G. In this paper, we study some properties of connected vertex-edge domination polynomials of the Friendship graph F_n. We obtain a recursive formula for $d_{cve}(F_n,i)$. Using this recursive formula, we construct the connected vertex - edge domination polynomial $D_{cve}(F_n,x) = \sum_{i=2}^{n+1} d_{cve}(F_n,i)\, x^i$ of F_n, where $d_{cve}(F_n,i)$ is the number of the connected vertex - edge dominating sets of F_n of cardinality i and some properties of this polynomial have been studied.

Keywords: *Friendship Graph, Connected Vertex - Edge Dominating Set, Connected Vertex - Edge, Domination Number, Connected Vertex - Edge Domination Polynomial.*

1. Introduction

Let $G = (V, E)$ be a simple graph of order n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set S $\subseteq V$, the open neighbouhood of S is

N(S) $=\bigcup_{v\in S} N(V)$ and the closed neighbourhood of S is $N[S] = N(s) \cup S$.

A vertex $u \in V(G)$ vertex - edge dominates (ve - dominates) an edge $vw \in E(G)$ if

1. $u = v$ or $u = w$ (u is incident to vw) or
2. uv or uw is an edge in G (u is incident to an edge that is adjacent to vw)

A vertex - edge dominating set S of G is called a connected vertex - edge dominating set if the induced subgraph $\langle S \rangle$ is connected.

The minimum cardinality of a connected vertex - edge dominating set of G is called the connected vertex - edge domination number of G and is denoted of $\gamma_{cve}(G)$. A connected vertex - edge dominating set with cardinality $\gamma_{cve}(G)$ is called $\gamma_{cve} -$ set.

Consider the Friendship graph F_n which has $n+1$ vertices. We use a recursive method to construct the families of connected vertex-edge dominating sets of F_n. The connected vertex – edge domination polynomials of the Friendship graph F_n are then studied. For the combination n to i we use $\binom{n}{i}$ as normal.

Connected vertex - edge dominating sets and connected vertex - edge domination polynomials of graphs

Definition 2.1: A set $S \subseteq V$ is a *dominating set* of G, if $N[S] = V$ or equivalently, every vertex in $V - S$ is adjacent to atleast one vertex in S. The *domination number* of a graph G is defined as the minimum cardinality taken over all dominating sets of vertices in G and it is denoted as $\gamma(G)$.

Definition 2.2: The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|v(G)|} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of cardinality i and $\gamma(G)$ is the domination number of G.

Definition 2.3: A vertex $u \in V(G)$ *vertex-edge dominates* (ve - dominates) an edge $vw \in E(G)$ if

(i) or $u = w$ (u is incident to vw) or

(ii) uv or uw is an edge in G (u is incident to an edge that is adjacent to uw).

Definition 2.4: A vertex-edge dominating set S of G is called a *connected vertex-edge dominating set* if the induced subgraph $< S >$ is connected.

Definition 2.5: The minimum cardinality of a connected vertex-edge dominating set of G is called the connected vertex-edge domination number of G and is denoted by $\gamma_{cve}(G)$. A connected vertex-edge dominating set with cardiality $\gamma_{cve}(G)$ is called γ_{cve} – set.

Definition 2.6: Let $D_{cve}(G,i)$ be the family of connected vertex-edge dominating set of G with cardinality i and let $d_{cve}(G,i) = |D_{cve}(G,i)|$.

Then the connected vertex-edge domination polynomial $D_{cve}(G,x)$ of G is defined as $D_{cve}(G,x) = \sum_{i=\gamma_{cve}}^{|v(G)|} d_{cve}(G,i)x^i$.

Example 2.7: Consider the graph G in the following Figure 1.27.

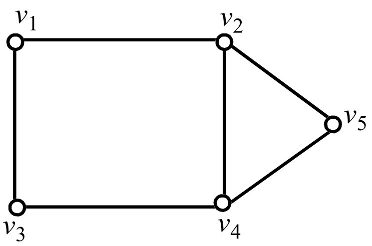
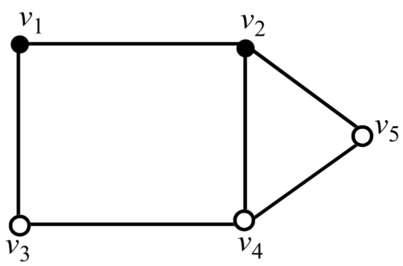


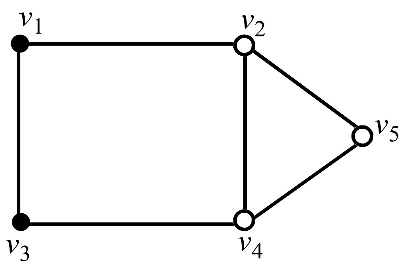
Figure 2.1

In Figure 1.28, S_1, S_2, S_3, S_4, S_5 and S_6 are the connected vertex-edge dominating sets of cardinalities 2.

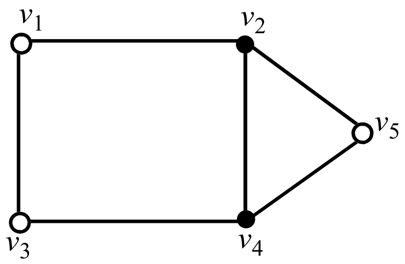
$S_1 = \{v_1, v_2\}$



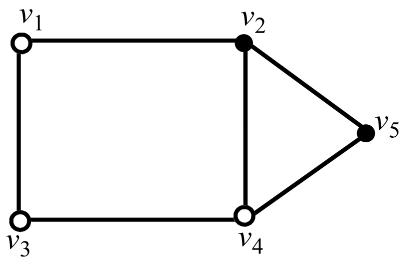
$S_2 = \{v_1, v_3\}$



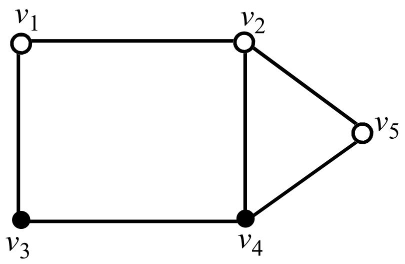
$S_3 = \{v_2, v_4\}$



$S_4 = \{v_2, v_5\}$



$S_3 = \{v_3, v_4\}$



$S_4 = \{v_4, v_5\}$

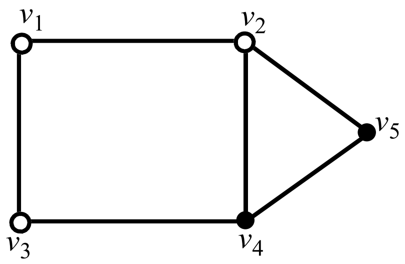
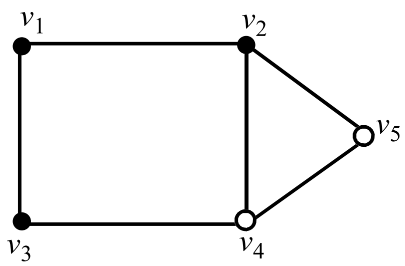


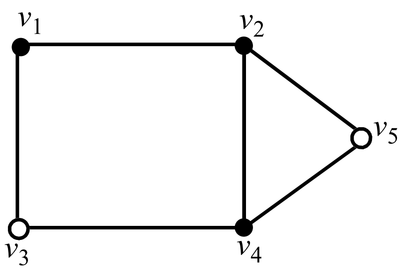
Figure 2.2

In Figure 1.29, $S_7, S_8, S_9, S_{10}, S_{11}, S_{12}$ and S_{13} are the connected vertex-edge dominating sets of cardinalities 3.

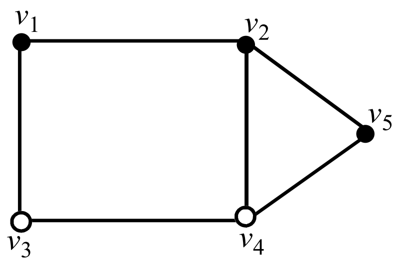
$S_7 = \{v_1, v_2, v_3\}$



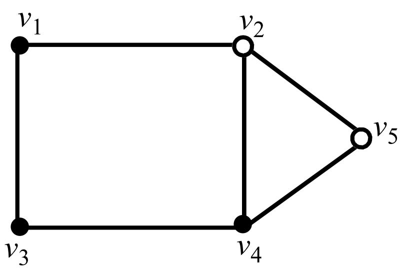
$S_8 = \{v_1, v_2, v_4\}$



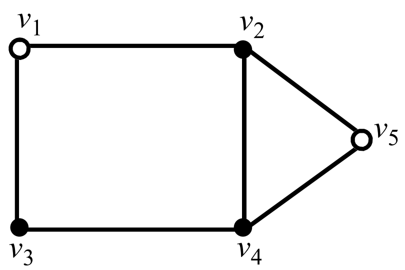
$S_9 = \{v_1, v_2, v_5\}$



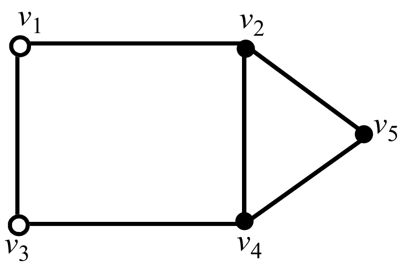
$S_{10} = \{v_1, v_3, v_4\}$



$S_{11} = \{v_2, v_3, v_4\}$



$S_{12} = \{v_2, v_4, v_5\}$



$S_{13} = \{v_3, v_4, v_5\}$

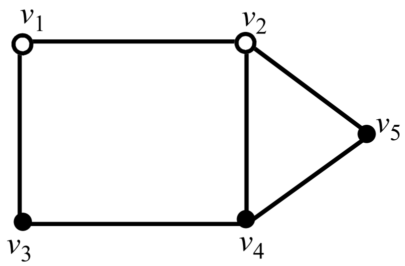
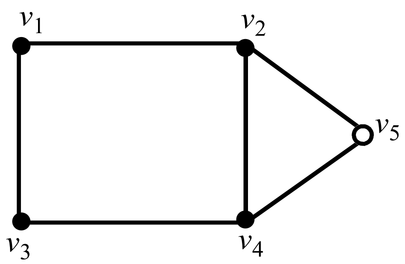


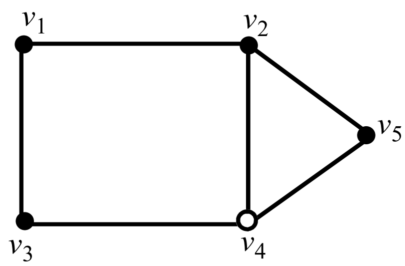
Figure 2.3

In Figure 1.30, $S_{14}, S_{15}, S_{16}, S_{17}$ and S_{18} are the connected vertex-edge dominating sets of cardinality 4.

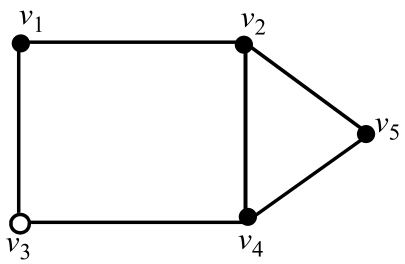
$S_{14} = \{v_1, v_2, v_3, v_4\}$



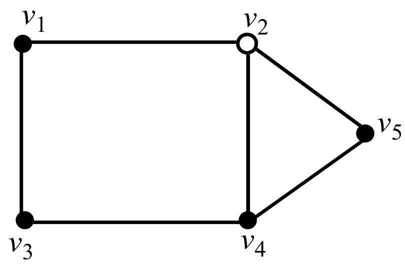
$S_{15} = \{v_1, v_2, v_3, v_5\}$



$S_{16} = \{v_1, v_2, v_4, v_5\}$



$S_{17} = \{v_1, v_3, v_4, v_5\}$



$S_{18} = \{v_2, v_3, v_4, v_5\}$

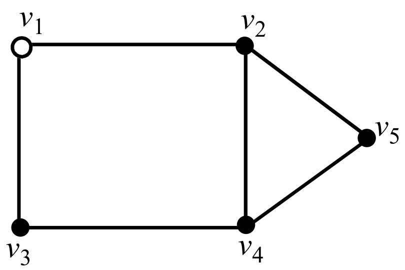


Figure 2.4

In Figure 1.30, S_{19} is the connected vertex-edge dominating sets of cardinality5.

$S_{19} = \{v_1, v_2, v_3, v_4, v_5\}$

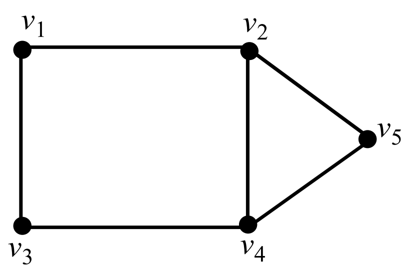


Figure 2.5

Here S_1, S_2, S_3, S_4 and S_5 are the minimum connected vertex-edge dominating sets.

Hence $\gamma_{cve}(G) = 2$.

The connected vertex-edge domination polynomial of G is

$$D_{cve}(G,x) = \sum_{i=\gamma_{cve}(G)}^{|V(G)|} d_{cve}(G,i)x^i$$

$$= \sum_{i=2}^{5} d_{cve}(G,i)x^i$$

$$= d_{cve}(G,2)x^2 + d_{cve}(G,3)x^3 + d_{cve}(G,4)x^4 + d_{cve}(G,5)x^5$$

$$= 5x^2 + 7x^3 + 5x^4 + x^5.$$

Connected Vertex – Edge Dominating Sets of Friendship Graph $\boldsymbol{F_n}$

Definition 3.1: The Friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex. It is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

Example 3.2: The Friendship graph F_4 is shown below:

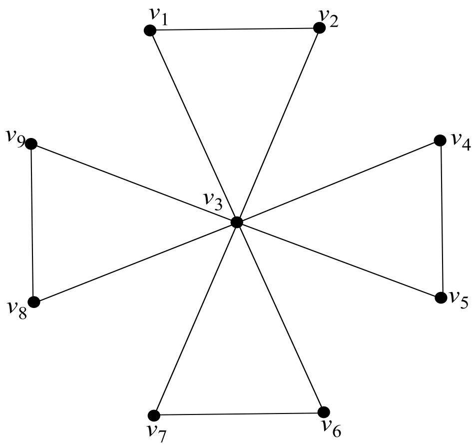


Figure 3.1

Definition 3.3: Let $D_{cve}(F_n, i)$ be the family of connected vertex – edge dominating sets of F_n with cardinality i. Then the connected vertex-edge domination number of F_n is defined as the minimum cardinality taken over all connected vertex – edge dominating sets of vertices in F_n and it is donoted by $\gamma_{cve}(F_n, i)$.

Lemma 3.4: Let F_n be the Friendship graph with $2n + 1$ vertices, then its connected vertex – edge domination number is $\gamma_{cve}(F_n) = 2$.

Proof: Let F_n be the Friendship graph with $2n + 1$ vertices and $3n$ edges. Let the vertices be $\{v_1, v_2, v_3, \dots, v_n, v_{n+1}, \dots, v_{2n}, v_{2n+1}\}$. It is the graph obtained by joining n copies of cycle C_3 with common vertex. Let the common vertex be v_3. For the n copies of cycle C_3 the two edges from each cycle must incident with the common vertex v_3. Also, by the definition of vertex – edge domination sets, all the edges are covered and they are connected. Thus, the minimum cardinality is 2. Hence, $\gamma_{cve}(F_n) = 2$.

Lemma 3.5: Let F_n, $n \geq 2$ be the Friendship graph with $|V(F_n)| = 2n + 1$. Then $d_{cve}(F_n, i) = 0$ if $i < 2$ or $i > n$ and $d_{cve}(F_n, i) > 0$ if $2 \leq i \leq n$.

Proof: If $i < 2$ or $i > n$, then there is no connected vertex - edge dominating set of cardinalities i. Therefore, $d_{cve}(F_n, i) = \varphi$. By lemma 2.4, the cardinality of the minimum connected vertex - edge dominating set is 2. Therefore, $d_{cve}(F_n, i) > 0$ if $i \geq 2$ and $i \leq n$. Hence, we have $d_{cve}(F_n, i) = 0$ if $i<2$ or $i > n$ and $d_{cve}(F_n, i) > 0$ if $2 \leq i \leq n$.

Lemma 3.6: Let F_n, $n \geq 2$ be the Friendship graph with $|V(F_n)| = 2n + 1$.

Then,

(i) $D_{cve}(F_n, x)$ has no constant and first-degree terms.

(ii) $D_{cve}(F_n, x)$ is a strictly increasing function on $[0,\infty)$.

Proof:

(i) Since the graph is a connected graph, at least two vertices must need cover the edges. So the polynomial $D_{cve}(F_n, x)$ has no constant and first degree terms.

(ii) Since n is increasing, the polynomial $D_{cve}(F_n, x)$ is strictly increasing on $[0,\infty)$.

Theorem 3.7: Let $F_n, n \geq 2$ be the Friendship graph with $2n + 1$ vertices, then

(i) $d_{cve}(F_n, i) = \binom{2n}{i-1}$ if $i \leq 2n + 1$.

(ii) $d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1}$ if $i < 2n + 1$.

Proof: Let F_n be a Friendship graph with $2n + 1$ vertices. Let the vertices be $v_1, v_2, v_{3,} \dots, v_n$, $v_{n+1}, \dots, v_{2n}, v_{2n+1}$. The Friendship graph F_n can be obtained by joining n copies of cycle C_3 with common vertex. Let the common vertex be v_3. There are $\binom{2n}{i-1}$ connected vertex - edge dominating sets with n vertices of cardinality i to need cover all the edges.Thus, $d_{cve}(F_n, i) = \binom{2n}{i-1}$ if $i \leq 2n + 1$.Also there are $\binom{2n-1}{i-1}$ connected vertex - edge dominating sets with $F_n - \{2n\}$ of cardinality i to need cover all the edges.

Thus, $d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1}$ if $i < 2n + 1$.Hence the proof is complete.

Theorem 3.8: Let $F_n, n \geq 2$ be the Friendship graph with $2n + 1$ vertices, then

(i) $d_{cve}(F_n, i) = d_{cve}(F_n - \{2n\}, i) + d_{cve}(F_n - \{2n\}, i - 1)$if $3 \leq i \leq n$.

(ii) $d_{cve}(F_n, i) = 1 + d_{cve}(F_n - \{2n\}, i)$ if $i = 2$.

(iii) $d_{cve}(F_n - \{2n\}, i) = d_{cve}(F_{n-1}, i) + d_{cve}(F_{n-1}, i - 1)$ if $3 \leq i \leq n$.

(iv) $d_{cve}(F_n - \{2n\}, i) = 1 + d_{cve}(F_{n-1}, i)$, if $i = 2$.

Proof: By Theorem 2.7, we have

$$d_{cve}(F_n, i) = \binom{2n}{i-1} \text{ if } i \leq 2n + 1 \text{ and}$$

$$d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1} \text{ if } i < 2n + 1.$$

(i) $d_{cve}(F_n - \{2n\}, i) + d_{cve}(F_n - \{2n\}, i - 1)$

$$= \binom{2n-1}{i-1} - \binom{2n-1}{i-2}$$

$$= \binom{2n}{i-1}$$

$$= d_{cve}(F_n, i)$$

(ii) Consider,

$d_{cve}(F_{n-1}, i) + d_{cve}(F_{n-1}, i - 1)$

$$= \binom{2(n-1)}{i-1} - \binom{2(n-1)}{i-1-1}$$

$$= \binom{2n-2}{i-1} + \binom{2n-2}{i-2}$$

$$= \binom{2n-1}{i-1}$$

$$= d_{cve}(F_n - \{2n\}, i)$$

Proof of (iii) and (iv) are obvious.

Connected vertex - edge domination polynomials of Friendship graph $\boldsymbol{F_n}$

Definition 4.1: Let $d_{cve}(F_n, i)$ be the number of connected vertex - edge dominating sets of the Lollipop graph F_n with cardinality i. Then the connected vertex - edge domination polynomial of F_n is defined as $D_{cve}(F_n, x) = \sum_{i=\gamma_{cve(F_n)}}^{2n+1} d_{cve}(F_n, i)\, x^i$, where $\gamma_{cve}(F_n)$ is the connected vertex – edge domination number of F_n.

Theorem 4.2: Let $D_{cve}(F_n, x)$ be the connected vertex - edge domination polynomial of a Friendship graph F_n with $2n+1$ vertices, then

(i) $D_{cve}(F_n, x) = \sum_{i=2}^{2n+1} \binom{2n}{i-1} x^i.$

(ii) $D_{cve}(F_n - \{2n\}, x) = \sum_{i=2}^{2n} \binom{2n-1}{i-1} x^i$

Proof: Proof follows from Theorem 3.7 and by the definition of connected vertex - edge domination polynomial.

Theorem 4.3: Let $D_{cve}(F_n, x)$ be the connected vertex – edge domination polynomial of a Friendship graph F_n with $2n+1$ vertices, then

(i) $D_{cve}(F_n, x) = x^2 + (1+x)D_{cve}(F_n - \{2n\}, x)$

(ii) $D_{cve}(F_n - \{2n\}, x) = x^2 + (1+x)D_{cve}(F_{n-1}, x).$

Proof: By the definition of connected vertex - edge domination polynomial, we have

$$D_{cve}(F_n, x) = \sum_{i=2}^{2n+1} d_{cve}(F_n, i)\, x^i$$

$$= d_{cve}(F_n, 2)x^2 + \sum_{i=3}^{2n+1} d_{cve}(F_n, i)\, x^i$$

$$= [1 + d_{cve}(F_n - \{2n\}, 2)]x^2 + \sum_{i=3}^{2n+1}\left[d_{cve}(F_n - \{2n\}, i)x^i\right]$$

$$+ \sum_{i=3}^{2n+1}\left[d_{cve}(F_n - \{2n\}, i - 1)x^i\right]$$

by Theorem 3.8

$$= x^2 + d_{cve}(F_n - \{2n\}, 2)x^2 + \sum_{i=3}^{2n+1} d_{cve}(F_n - \{2n\}, i)\, x^i$$

$$+ \sum_{i=3}^{2n+1} d_{cve}(F_n - \{2n\}, i-1)\, x^i$$

$$= x^2 + \sum_{i=2}^{2n+1} d_{cve}(F_n - \{2n\}, i)\, x^i + x\sum_{i=3}^{2n+1} d_{cve}(F_n - \{2n\}, i-1)\, x^{i-1}$$

$$= x^2 + D_{cve}(F_n - \{2n\}, x) + xD_{cve}(F_n - \{2n\}, x)$$

$$= x^2 + (1+x)D_{cve}(F_n - \{2n\}, x).$$

(ii) By the definition of connected vertex - edge domination polynomial, we have

$$D_{cve}(F_n - \{2n\}, x) = \sum_{i=2}^{2n+1} d_{cve}(F_n - \{2n\}, i)\, x^i$$

$$= d_{cve}(F_n - \{2n\}, 2)x^2 + \sum_{i=3}^{2n+1} d_{cve}(F_n - \{2n\}, i)\, x^i$$

$$= [1 + d_{cve}(F_{n-1}, 2)]x^2 + \sum_{i=3}^{2n+1}[d_{cve}(F_{n-1}, i) + x^i\, d_{cve}(F_{n-1}, i-1)]$$

$$= x^2 + d_{cve}(F_{n-1}, 2)x^2 + \sum_{i=3}^{2n+1} d_{cve}(F_{n-1}, i)\, x^i$$

$$+ \sum_{i=3}^{2n+1} d_{cve}(F_{n-1}, i-1)\, x^i$$

$$= x^2 + \sum_{i=2}^{2n+1} d_{cve}(F_{n-1}, i)\, x^i + x\sum_{i=3}^{2n+1} d_{cve}(F_{n-1}, i-1)\, x^{i-1}$$

$$= x^2 + D_{cve}(F_{n-1}, x) + xD_{cve}(F_{n-1}, x)$$

$= x^2 + (1+x)D_{cve}(F_{n-1}, x).$

Example 4.4: $D_{ve}(F_4, x) = 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9$

Verification: By Theorem 4.3, we have

$$D_{cve}(F_4, x) = x^2 + (1+x)D_{cve}(F_4 - \{8\}, x)$$

$$= x^2 + (1+x)[7x^2 + 21x^3 + 35x^4 + 35x^5 + 21x^6 + 7x^7 + x^8]$$

$$= x^2 + 7x^2 + 21x^3 + 35x^4 + 35x^5 + 21x^6 + 7x^7 + x^8 + 7x^3 + 21x^4 + 35x^5 + 35x^6 + 21x^7 + 7x^8 + x^9$$

$$= 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9$$

Example: $D_{cve}(F_5 - \{10\}, x) = 8x^2 + 36x^3 + 84x^4 + 126x^5 + 126x^6 + 84x^7 + 36x^8 + 9x^9 + x^{10}$

Verification: By Theorem 4.3, we have

$$D_{cve}(F_5 - \{10\}, x) = x^2 + (1+x)D_{cve}(F_4, x)$$

$$= x^2 + (1+x)[8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9]$$

$$= x^2 + 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9 + 8x^3 + 28x^4 + 56x^5 + 70x^6 + 56x^7 + 28x^8 + 8x^9 + x^{10}$$

$$= 8x^2 + 36x^3 + 84x^4 + 126x^5 + 126x^6 + 84x^7 + 36x^8 + 9x^9 + x^{10}$$

We obtain $d_{cve}(F_n, i)$, $2 \le n \le 15$ as shown in the following table 1:

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_2 - \{4\}$	4	3	1											
F_2	4	6	4	1										
$F_2 - \{6\}$	5	10	10	5	1									
F_3	6	15	20	15	6	1								
$F_2 - \{8\}$	7	21	35	35	21	7	1							
F_4	8	28	56	70	56	28	8	1						
9	9	36	84	126	126	84	36	9	1					
F_5	10	45	120	210	252	210	120	45	10	1				
$F_2 - \{12\}$	11	55	165	330	462	462	330	165	55	11	1			
F_6	12	66	220	495	792	924	792	495	220	66	12	1		
$F_2 - \{14\}$	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
F_7	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1

Theorem 4.5: For every $n \in N$ and $3 \le i \le n$, $|D_{cve}(L_{n,1}, i)|$ is the coefficient of $u^n v^i$ in the expansion of the function $f(u,v) = \frac{u^4 v^3 [(v+2)^2+3}{1-u(1+v)}$

Proof: Set $f(u,v) = \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)| u^n v^i$ by recursive formula for $|D_{cve}(L_{n,1}, i)|$ in Theorem 2-9 we can write $f(u,v)$ in the following form:

$$f(u,v) = \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} [|D_{cve}(L_{n-1,1}, i-1)| + |D_{cve}(L_{n-1,1}, i)| u^n v^i$$

$$= \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n-1,1}, i-1)| u^n v^i + \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n-1,1}, i)| u^n v^i$$

$$= uv \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n-1,1}, i-1)| u^{n-1} v^{i-1} + u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n-1,1}, i)| u^{n-1} v^i$$

$$= uv \Big[[|D_{cve}(L_{3,1}, 2)| u^3 v^2 + |D_{cve}(L_{3,1}, 3)| u^3 v^3 + |D_{cve}(L_{3,1} 4)| u^3 v^4 \Big]$$

$$+ uv \sum_{n=5}^{\infty} \sum_{i=2}^{\infty} |D_{cve}(L_{n-1,1}, i-1)| u^{n-1} v^{i-1} \Big]$$

$$+ u\big[|D_{cve}(L_{3,1}, 3)| u^3 v^3 + |D_{cve}(L_{3,1}, 4)| u^3 v^4 + u \sum_{n=5}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n-1,1}, i)| u^{n-1} v^i \big].$$

$D_{cve}(L_{n,1}, i)$ is family of connected vertex-edge dominating set with cardinality i of $L_{n,1}$.

From Table 1, we have, $|D_{cve}(L_{3,1}, 2)| = 4$, $|D_{cve}(L_{3,1}, 3)| = 3$and $|D_{cve}(L_{3,1}, 4)| = 1$.

Then $f(u,v) = uv[4u^3v^2 + 3u^3v^3 + u^3v^4] + uvf(u,v) + u[3u^3v^3 + u^3v^4] + uf(u,v)$

$$= uvu^3v^2[4 + 3v + v^2] + uvf(u,v) + u\,u^3v^3[3 + v] + uf(u,v)$$

$$f(u,v) = u^4v^3[4 + 3v + v^2] + u^4v^3[3 + v] + f(u,v)[uv + u]$$

$$f(u,v) = u^4v^3[v^2 + 3v + 4] + u^4v^3[3 + v] + f(u,v)[uv + u]$$

$$= u^4v^3[v^2 + 3v + 4 + 3 + v] + f(u,v)[uv + u]$$

$$= u^4v^3[v^2 + 4v + 7] + f(u,v)[uv + u]$$

$$f(u,v) - f(u,v)[uv + v] = u^4v^3[v^2 + 4v + 7]$$

$$f(u,v)[1 - uv - vu] = u^4v^3[v^2 + 4v + 4 + 3]$$

$$f(u,v)[1 - u(1 + v)] = u^4v^3[(v + 2)^2 + 3]$$

Hence, $f(u,v) = \frac{u^4v^3[(v+2)^2+3]}{1-u(1+v)}$.

4. Conclusion

In this paper, the connected vertex - edge domination polynomials of Friendship graph F_n has been derived by identifying its connected vertex - edge dominating sets. Also find the recursive formula for connected vertex - edge dominating sets and using this relation I have derived some interesting properties.

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References:

1. S. Alikani and Y.H. Peng, 2008. "Dominating sets and domination polynomials of cycles", Global Journal of Pure and Applied Mathematics, Vol. 4 No.2.
2. S. Alikani and Y.H Peng, 2005. "Introduction to domination polynomial of a graph",
3. arXiv : 0905.225 Vol.10, No. 14.
4. G. Chartrand and P. Zhamg," Introduction to graph theory", McGraw- Hill, Boston, Mass, USA.
5. Vijayan,A & Nagarajan, T 2013. "Vertex - Edge dominating sets and vertex-edge domination polynomials of Paths", International Journal of Mathematics Trends and Technology, Vol.4(11) , 266-279.
6. Radhika, VS & Vijayan, A, 2021. "Connected vertex-edge dominating sets and connected vertex- edge domination polynomials of triangular ladder', Malaya Journal of Mathematik, Vol 9, No.1, 474-479.
7. Radhika, VS & Vijayan, A, 2021. "Connected vertex-edge domination polynomials of some graphs", Advances and Applications in Discrete Mathematics, Vol.27, No.2, 183-192.
8. Radhika, VS & Vijayan, A, 2021. "Connected Vertex-edge dominating sets and connected vertex-edge domination polynomials of gem graph G_n", Journal of Physics: Conference Series: Vol.1947 012044.
9. Radhika, VS & Vijayan, A, 2020, "Connected vertex-edge dominating sets and connected domination polynomials of square of paths", Adalya, Vol.9, 15-25.

10. Radhika, VS & Vijayan, A, 2021, “connected vertex-edge domination polynomials of fan graph $F_{2,n}$”, AIP Conference Proceedings.