Connected Vertex-Edge Dominating Sets and Connected Vertex-Edge Domination Polynomials of Friendship $F_n$

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<th>Article History</th>
<th>Abstract</th>
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<tr>
<td>Received: 06 June 2023</td>
<td>Let $G$ be a simple connected graph of order $n$. Let $D_{cve}(G, i)$ be the family of connected vertex-edge dominating sets in $G$ with cardinality $i$. The polynomial $D_{cve}(G, x) = \sum_{i=0}^{n} d_{cve}(G, i) x^i$ is called the connected vertex-edge domination polynomial of $G$, where $d_{cve}(G, i)$ is the number of connected vertex-edge dominating sets of $G$. In this paper, we study some properties of connected vertex-edge domination polynomials of the Friendship graph $F_n$. We obtain a recursive formula for $d_{cve}(F_n, i)$. Using this recursive formula, we construct the connected vertex-edge domination polynomial $D_{cve}(F_n, x) = \sum_{i=0}^{n+1} d_{cve}(F_n, i) x^i$ of $F_n$, where $d_{cve}(F_n, i)$ is the number of the connected vertex-edge dominating sets of $F_n$ of cardinality $i$ and some properties of this polynomial have been studied.</td>
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<td><strong>Keywords:</strong> Friendship Graph, Connected Vertex - Edge Dominating Set, Connected Vertex - Edge, Domination Number, Connected Vertex - Edge Domination Polynomial.</td>
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### 1. Introduction

Let $G = (V, E)$ be a simple graph of order $n$. For any vertex $v \in V$, the open neighbourhood of $v$ is the set $N(v) = \{ u \in V \mid uv \in E \}$ and the closed neighbourhood of $v$ is the set $N[v] = N(v) \cup \{ v \}$. For a set $S \subseteq V$, the open neighbourhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of $S$ is $N[S] = N(S) \cup S$.

A vertex $u \in V(G)$ vertex-edge dominates (ve-dominates) an edge $vw \in E(G)$ if

1. $u = v$ or $u = w$ ($u$ is incident to $vw$) or
2. $uw$ or $uw$ is an edge in $G$ ($u$ is incident to an edge that is adjacent to $vw$)

A vertex-edge dominating set $S$ of $G$ is called a connected vertex-edge dominating set if the induced subgraph $(S)$ is connected.

The minimum cardinality of a connected vertex-edge dominating set of $G$ is called the connected vertex-edge domination number of $G$ and is denoted by $\gamma_{cve}(G)$. A connected vertex-edge dominating set with cardinality $\gamma_{cve}(G)$ is called $\gamma_{cve} -$ set.

Consider the Friendship graph $F_n$ which has $n + 1$ vertices. We use a recursive method to construct the families of connected vertex-edge dominating sets of $F_n$. The connected vertex-edge domination polynomials of the Friendship graph $F_n$ are then studied. For the combination $n$ to $i$ we use $\binom{n}{i}$ as normal.
Definition 2.1: A set $S \subseteq V$ is a dominating set of $G$, if $N[S] = V$ or equivalently, every vertex in $V - S$ is adjacent to at least one vertex in $S$. The domination number of a graph $G$ is defined as the minimum cardinality taken over all dominating sets of vertices in $G$ and it is denoted as $\gamma(G)$.

Definition 2.2: The domination polynomial $D(G, x)$ of $G$ is defined as $D(G, x) = \Sigma_{i=\gamma(G)}^{\vert v(G)\vert} d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of $G$ of cardinality $i$ and $\gamma(G)$ is the domination number of $G$.

Definition 2.3: A vertex $u \in V(G)$ vertex-edge dominates (ve dominates) an edge $vw \in E(G)$ if

(i) or $u = w$ (u is incident to vw)
(ii) $uv$ or $uw$ is an edge in $G$ (u is incident to an edge that is adjacent to uw).

Definition 2.4: A vertex-edge dominating set $S$ of $G$ is called a connected vertex-edge dominating set if the induced subgraph $\langle S \rangle$ is connected.

Definition 2.5: The minimum cardinality of a connected vertex-edge dominating set of $G$ is called the connected vertex-edge domination number of $G$ and is denoted by $\gamma_{cve}(G)$. A connected vertex-edge dominating set with cardinality $\gamma_{cve}(G)$ is called $\gamma_{cve}$-set.

Definition 2.6: Let $D_{cve}(G, i)$ be the family of connected vertex-edge dominating set of $G$ with cardinality $i$ and let $d_{cve}(G, i) = \vert D_{cve}(G, i) \vert$. Then the connected vertex-edge domination polynomial $D_{cve}(G, x)$ of $G$ is defined as $D_{cve}(G, x) = \Sigma_{i=\gamma_{cve}(G)}^{\vert v(G)\vert} d_{cve}(G, i)x^i$.

Example 2.7: Consider the graph $G$ in the following Figure 1.27.

Figure 2.1

In Figure 1.28, $S_1, S_2, S_3, S_4, S_5$ and $S_6$ are the connected vertex-edge dominating sets of cardinalities 2.

$S_1 = \{v_1, v_2\}$

$S_2 = \{v_1, v_3\}$
In Figure 1.29, $S_7, S_8, S_9, S_{10}, S_{11}, S_{12}$ and $S_{13}$ are the connected vertex-edge dominating sets of cardinalities 3.

Figure 2.2

- $S_3 = \{v_2, v_4\}$
- $S_4 = \{v_2, v_3\}$
- $S_3 = \{v_3, v_4\}$
- $S_4 = \{v_4, v_5\}$
- $S_7 = \{v_1, v_2, v_3\}$
- $S_8 = \{v_1, v_2, v_4\}$
- $S_9 = \{v_1, v_2, v_5\}$
- $S_{10} = \{v_1, v_3, v_4\}$
- $S_{11} = \{v_2, v_3, v_4\}$
- $S_{12} = \{v_2, v_4, v_5\}$
$S_{13} = \{v_3, v_4, v_5\}$

**Figure 2.3**

In Figure 1.30, $S_{14}, S_{15}, S_{16}, S_{17}$ and $S_{18}$ are the connected vertex-edge dominating sets of cardinality 4.

$S_{14} = \{v_1, v_2, v_3, v_4\}$

$S_{15} = \{v_1, v_2, v_3, v_5\}$

$S_{16} = \{v_1, v_2, v_4, v_5\}$

$S_{17} = \{v_1, v_3, v_4, v_5\}$

$S_{18} = \{v_2, v_3, v_4, v_5\}$

**Figure 2.4**

In Figure 1.30, $S_{19}$ is the connected vertex-edge dominating sets of cardinality 5.

$S_{19} = \{v_1, v_2, v_3, v_4, v_5\}$

**Figure 2.5**
Here $S_1, S_2, S_3, S_4$ and $S_5$ are the minimum connected vertex-edge dominating sets.

Hence $\gamma_{cv}(G) = 2$.

The connected vertex-edge domination polynomial of $G$ is

$$D_{cv}(G, x) = \sum_{i=\gamma_{cv}(G)}^{\mid V(G) \mid} d_{cv}(G, i)x^i$$

$$= \sum_{i=2}^{5} d_{cv}(G, i)x^i$$

$$= d_{cv}(G, 2)x^2 + d_{cv}(G, 3)x^3 + d_{cv}(G, 4)x^4 + d_{cv}(G, 5)x^5$$

$$= 5x^2 + 7x^3 + 5x^4 + x^5.$$ 

**Connected Vertex – Edge Dominating Sets of Friendship Graph $F_n$**

**Definition 3.1:** The Friendship graph $F_n$ can be constructed by joining $n$ copies of the cycle graph $C_3$ with a common vertex. It is a planar undirected graph with $2n + 1$ vertices and $3n$ edges.

**Example 3.2:** The Friendship graph $F_4$ is shown below:

![Figure 3.1](image-url)

**Definition 3.3:** Let $D_{cv}(F_n, i)$ be the family of connected vertex–edge dominating sets of $F_n$ with cardinality $i$. Then the connected vertex-edge domination number of $F_n$ is defined as the minimum cardinality taken over all connected vertex–edge dominating sets of vertices in $F_n$ and it is denoted by $\gamma_{cv}(F_n, i)$.

**Lemma 3.4:** Let $F_n$ be the Friendship graph with $2n + 1$ vertices, then its connected vertex–edge domination number is $\gamma_{cv}(F_n) = 2$.

**Proof:** Let $F_n$ be the Friendship graph with $2n + 1$ vertices and $3n$ edges. Let the vertices be \{ $v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}$\}. It is the graph obtained by joining $n$ copies of cycle $C_3$ with a common vertex. Let the common vertex be $v_3$. For the $n$ copies of cycle $C_3$ the two edges from each cycle must incident with the common vertex $v_3$. Also, by the definition of vertex – edge domination sets, all the edges are covered and they are connected. Thus, the minimum cardinality is 2. Hence, $\gamma_{cv}(F_n) = 2$.

**Lemma 3.5:** Let $F_n$, $n \geq 2$ be the Friendship graph with $|V(F_n)| = 2n + 1$. Then $d_{cv}(F_n, i) = 0$ if $i < 2$ or $i > n$ and $d_{cv}(F_n, i) > 0$ if $2 \leq i \leq n$.

**Proof:** If $i < 2$ or $i > n$, then there is no connected vertex - edge dominating set of cardinalities $i$. Therefore, $d_{cv}(F_n, i) = \varphi$. By lemma 2.4, the cardinality of the minimum connected vertex - edge dominating set is 2. Therefore, $d_{cv}(F_n, i) > 0$ if $i \geq 2$ and $i \leq n$. Hence, we have $d_{cv}(F_n, i) = 0$ if $i < 2$ or $i > n$ and $d_{cv}(F_n, i) > 0$ if $2 \leq i \leq n$.

**Lemma 3.6:** Let $F_n$, $n \geq 2$ be the Friendship graph with $|V(F_n)| = 2n + 1$.

Then,

(i) $D_{cv}(F_n, x)$ has no constant and first-degree terms.
(ii) $D_{cve}(F_n, x)$ is a strictly increasing function on $[0, \infty)$.

Proof:

(i) Since the graph is a connected graph, at least two vertices must need cover the edges. So the polynomial $D_{cve}(F_n, x)$ has no constant and first degree terms.

(ii) Since $n$ is increasing, the polynomial $D_{cve}(F_n, x)$ is strictly increasing on $[0, \infty)$.

Theorem 3.7: Let $F_n, n \geq 2$ be the Friendship graph with $2n + 1$ vertices, then

(i) $d_{cve}(F_n, i) = \binom{2n}{i-1}$ if $i \leq 2n + 1$.

(ii) $d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1}$ if $i < 2n + 1$.

Proof: Let $F_n$ be a Friendship graph with $2n + 1$ vertices. Let the vertices be $v_1, v_2, v_3, ..., v_n, v_{n+1}, ..., v_{2n}, v_{2n+1}$. The Friendship graph $F_n$ can be obtained by joining $n$ copies of cycle $C_3$ with common vertex. Let the common vertex be $v_3$. There are $\binom{2n}{i-1}$ connected vertex - edge dominating sets with $n$ vertices of cardinality $i$ to need cover all the edges. Thus, $d_{cve}(F_n, i) = \binom{2n}{i-1}$ if $i \leq 2n + 1$. Also there are $\binom{2n-1}{i-1}$ connected vertex - edge dominating sets with $F_n - \{2n\}$ of cardinality $i$ to need cover all the edges.

Thus, $d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1}$ if $i < 2n + 1$. Hence the proof is complete.

Theorem 3.8: Let $F_n, n \geq 2$ be the Friendship graph with $2n + 1$ vertices, then

(i) $d_{cve}(F_n, i) = d_{cve}(F_n - \{2n\}, i) + d_{cve}(F_n - \{2n\}, i - 1)\text{ if } 3 \leq i \leq n$.

(ii) $d_{cve}(F_n, i) = 1 + d_{cve}(F_n - \{2n\}, i)\text{ if } i = 2$.

(iii) $d_{cve}(F_n - \{2n\}, i) = d_{cve}(F_{n-1}, i) + d_{cve}(F_{n-1}, i - 1)\text{ if } 3 \leq i \leq n$.

(iv) $d_{cve}(F_n - \{2n\}, i) = 1 + d_{cve}(F_{n-1}, i),\text{ if } i = 2$.

Proof: By Theorem 2.7, we have

\[
d_{cve}(F_n, i) = \binom{2n}{i-1}\text{ if } i \leq 2n + 1\text{ and}
d_{cve}(F_n - \{2n\}, i) = \binom{2n-1}{i-1}\text{ if } i < 2n + 1.
\]

(i) $d_{cve}(F_n - \{2n\}, i) + d_{cve}(F_n - \{2n\}, i - 1)$

\[
= \binom{2n-1}{i-1} - \binom{2n-1}{i-2}
= \binom{2n}{i-1}
= d_{cve}(F_n, i)
\]

(ii) Consider,

\[
d_{cve}(F_{n-1}, i) + d_{cve}(F_{n-1}, i - 1)
= \binom{2(n-1)}{i-1} - \binom{2(n-1)}{i-1-1}
= \binom{2n-2}{i-1} + \binom{2n-2}{i-2}
= \binom{2n-1}{i-1}
\]
\[ = d_{\text{cve}}(F_n - \{2n\}, i) \]

Proof of (iii) and (iv) are obvious.

**Connected vertex - edge domination polynomials of Friendship graph** \( F_n \)

**Definition 4.1:** Let \( d_{\text{cve}}(F_n, i) \) be the number of connected vertex - edge dominating sets of the Lollipop graph \( F_n \) with cardinality \( i \). Then the connected vertex - edge domination polynomial of \( F_n \) is defined as \( D_{\text{cve}}(F_n, x) = \sum_{i=\gamma_{\text{cve}(F_n)}}^{2n+1} d_{\text{cve}}(F_n, i) x^i \), where \( \gamma_{\text{cve}(F_n)} \) is the connected vertex - edge domination number of \( F_n \).

**Theorem 4.2:** Let \( D_{\text{cve}}(F_n, x) \) be the connected vertex - edge domination polynomial of a Friendship graph \( F_n \) with \( 2n + 1 \) vertices, then

(i) \[ D_{\text{cve}}(F_n, x) = \sum_{i=2}^{2n+1} \binom{2n}{i-1} x^i. \]

(ii) \[ D_{\text{cve}}(F_n - \{2n\}, x) = \sum_{i=2}^{2n} \binom{2n-1}{i-1} x^i \]

**Proof:** Proof follows from Theorem 3.7 and by the definition of connected vertex - edge domination polynomial.

**Theorem 4.3:** Let \( D_{\text{cve}}(F_n, x) \) be the connected vertex – edge domination polynomial of a Friendship graph \( F_n \) with \( 2n + 1 \) vertices, then

(i) \[ D_{\text{cve}}(F_n, x) = x^2 + (1 + x)D_{\text{cve}}(F_n - \{2n\}, x) \]

(ii) \[ D_{\text{cve}}(F_n - \{2n\}, x) = x^2 + (1 + x)D_{\text{cve}}(F_{n-1}, x). \]

**Proof:** By the definition of connected vertex - edge domination polynomial, we have

\[ D_{\text{cve}}(F_n, x) = \sum_{i=2}^{2n+1} d_{\text{cve}}(F_n, i) x^i \]

\[ = d_{\text{cve}}(F_n, 2) x^2 + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_n, i) x^i \]

\[ = [1 + d_{\text{cve}}(F_n - \{2n\}, 2)]x^2 + \sum_{i=3}^{2n+1} \left[ d_{\text{cve}}(F_n - \{2n\}, i)x^i \right] \]

\[ + \sum_{i=3}^{2n+1} \left[ d_{\text{cve}}(F_n - \{2n\}, i-1) \right] \]

by Theorem 3.8

\[ = x^2 + d_{\text{cve}}(F_n - \{2n\}, 2)x^2 + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i)x^i \]

\[ + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i-1)x^i \]

\[ = x^2 + \sum_{i=2}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i)x^i + x \sum_{i=3}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i-1)x^{i-1} \]

\[ = x^2 + D_{\text{cve}}(F_n - \{2n\}, x) + xD_{\text{cve}}(F_n - \{2n\}, x) \]

\[ = x^2 + (1 + x)D_{\text{cve}}(F_n - \{2n\}, x). \]

(ii) By the definition of connected vertex - edge domination polynomial, we have

\[ D_{\text{cve}}(F_n - \{2n\}, x) = \sum_{i=2}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i)x^i \]

\[ = d_{\text{cve}}(F_n - \{2n\}, 2)x^2 + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_n - \{2n\}, i)x^i \]

\[ = [1 + d_{\text{cve}}(F_{n-1}, 2)]x^2 + \sum_{i=3}^{2n+1} [d_{\text{cve}}(F_{n-1}, i) + x^i d_{\text{cve}}(F_{n-1}, i-1)] \]

\[ = x^2 + d_{\text{cve}}(F_{n-1}, 2)x^2 + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_{n-1}, i)x^i \]

\[ + \sum_{i=3}^{2n+1} d_{\text{cve}}(F_{n-1}, i-1)x^i \]

\[ = x^2 + \sum_{i=2}^{2n+1} d_{\text{cve}}(F_{n-1}, i)x^i + x \sum_{i=3}^{2n+1} d_{\text{cve}}(F_{n-1}, i-1)x^{i-1} \]

\[ = x^2 + D_{\text{cve}}(F_{n-1}, x) + xD_{\text{cve}}(F_{n-1}, x). \]
\( = x^2 + (1 + x)D_{cve}(F_{n-1}, x). \)

**Example 4.4:** \( D_{cve}(F_4, x) = 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9 \)

**Verification:** By Theorem 4.3, we have
\[
D_{cve}(F_4, x) = x^2 + (1 + x)D_{cve}(F_4 - \{8\}, x) \]
\[
= x^2 + (1 + x)[7x^2 + 21x^3 + 35x^4 + 35x^5 + 21x^6 + 7x^7 + x^8] \]
\[
= x^2 + 7x^2 + 21x^3 + 35x^4 + 35x^5 + 21x^6 + 7x^7 + x^8 + \frac{35x^5}{35x^6 + 21x^7 + 7x^8 + x^9} \]
\[
= 8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9 \]

**Example:** \( D_{cve}(F_5 - \{10\}, x) = 8x^2 + 36x^3 + 84x^4 + 126x^5 + 126x^6 + 84x^7 + 36x^8 + 9x^9 + x^{10} \)

**Verification:** By Theorem 4.3, we have
\[
D_{cve}(F_5 - \{10\}, x) = x^2 + (1 + x)D_{cve}(F_4, x) \]
\[
= x^2 + (1 + x)[8x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9]\]
\[
= x^2 + 28x^3 + 56x^4 + 70x^5 + 56x^6 + 28x^7 + 8x^8 + x^9]
\[
= 8x^2 + 36x^3 + 84x^4 + 126x^5 + 126x^6 + 84x^7 + 36x^8 + 9x^9 + x^{10} \]

We obtain \( d_{cve}(F_n, i), 2 \leq n \leq 15 \) as shown in the following table 1:

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<tr>
<td>( F_7 )</td>
<td>1</td>
<td>9</td>
<td>1</td>
<td>364</td>
<td>100</td>
<td>1</td>
<td>2002</td>
<td>3003</td>
<td>343</td>
<td>2</td>
<td>300</td>
<td>3</td>
<td>200</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>
Theorem 4.5: For every \( n \in N \) and \( 3 \leq i \leq n \), \( |D_{cve}(L_{n,1}, i)| \) is the coefficient of \( u^n v^i \) in the expansion of the function \( f(u, v) = \frac{u^4 v^3[4 + 3v + 2v^2] + u^4 v^3[3 + v] + f(u, v)[uv + u]}{1 - u(1 + v)} \)

Proof: Set \( f(u, v) = \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)|u^n v^i \) by recursive formula for \( |D_{cve}(L_{n,1}, i)| \) in Theorem 2.9 we can write \( f(u, v) \) in the following form:

\[
f(u, v) = \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i - 1)|u^n v^i + \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)|u^n v^i
\]

\[
= \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i - 1)|u^n v^i + \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)|u^n v^i
\]

\[
= u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i - 1)|u^n v^i + u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)|u^n v^i
\]

\[
= u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, 2)|u^n v^2 + |D_{cve}(L_{n,1}, 3)|u^n v^3 + |D_{cve}(L_{n,1}, 4)|u^n v^4
\]

\[
+ u \sum_{n=4}^{\infty} \sum_{i=2}^{\infty} |D_{cve}(L_{n,1}, i - 1)|u^n v^i-1
\]

From Table 1, we have, \( |D_{cve}(L_{n,1}, 2)| = 4, |D_{cve}(L_{n,1}, 3)| = 3 \) and \( |D_{cve}(L_{n,1}, 4)| = 1 \).

Then \( f(u, v) = u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i - 1)|u^n v^i + u \sum_{n=4}^{\infty} \sum_{i=3}^{\infty} |D_{cve}(L_{n,1}, i)|u^n v^i \)

\[
f(u, v) = u^4 v^3[4 + 3v + 2v^2] + u^4 v^3[3 + v] + f(u, v)[uv + u]
\]

\[
f(u, v) = u^4 v^3[2v^2 + 3v + 4] + u^4 v^3[3 + v] + f(u, v)[uv + u]
\]

\[
= u^4 v^3[2v^2 + 3v + 4 + 3 + v] + f(u, v)[uv + u]
\]

\[
= u^4 v^3[2v^2 + 4v + 7] + f(u, v)[uv + u]
\]

\[
f(u, v) = u^4 v^3[2v^2 + 4v + 7]
\]

\[
f(u, v)[1 - uv - vu] = u^4 v^3[2v^2 + 4v + 4 + 3]
\]

\[
f(u, v)[1 - u(1 + v)] = u^4 v^3[(v + 2)^2 + 3]
\]

Hence, \( f(u, v) = \frac{u^4 v^3[(v + 2)^2 + 3]}{1 - u(1 + v)} \).

4. Conclusion

In this paper, the connected vertex - edge domination polynomials of Friendship graph \( F_n \) has been derived by identifying its connected vertex - edge dominating sets. Also find the recursive formula for connected vertex - edge dominating sets and using this relation I have derived some interesting properties.

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